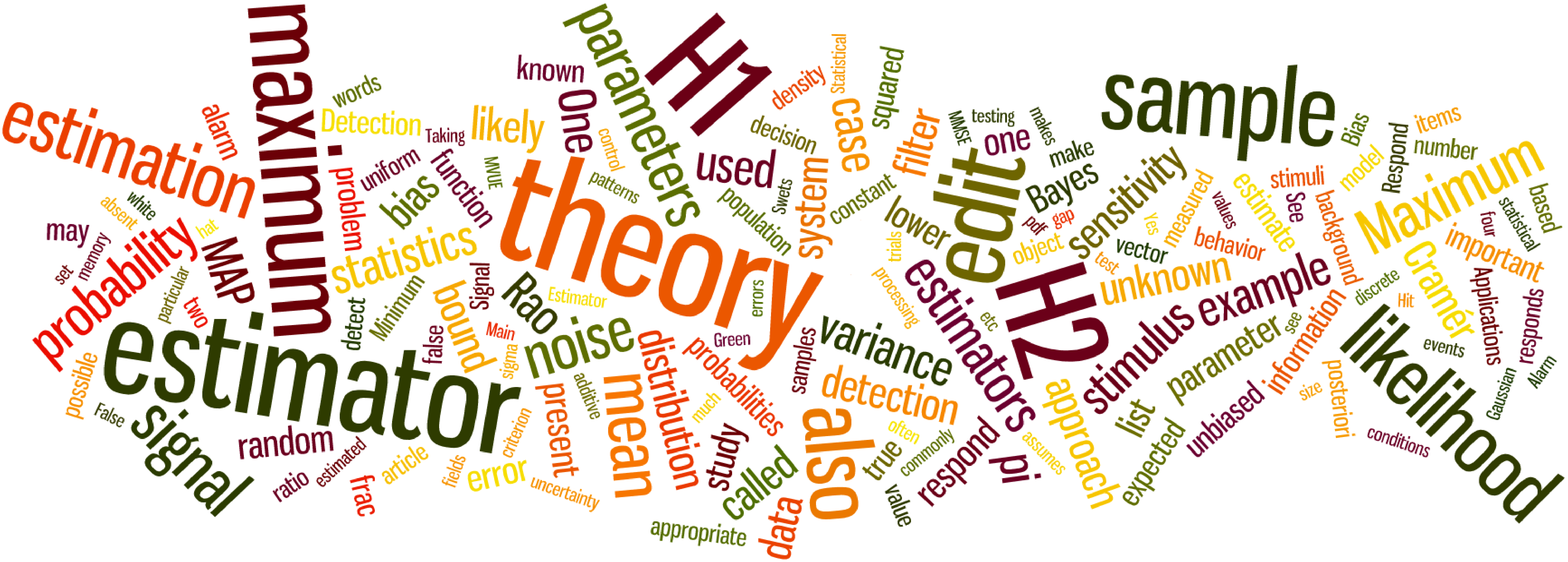

Detection and Estimation Theory
Lecture 9

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Based on ECE 531 Slides- 2011 (Prof. Natasha Devroye)

Finding MVUE- so far

1. CRLB - may give you the MVUE
2. Linear models - MVUE and its statistics explicitly!
3. Rao-Blackwell-Lehmann-Scheffe (RBLS) theorem - may give you the MVUE if you can find sufficient and complete statistics

MVUE still may be tough to find

Possible issues include: (i) knowledge of the PDF (ii) data model.

Finding MVUE- so far

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Possible issues include: (i) knowledge of the PDF (ii) data model.

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A “Linear Estimator” may promise a solution by only requiring first and second order moments of the PDF.

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Fairly practical!

Best Linear Unbiased Estimator (BLUE)

- It simplifies finding an estimator by constraining the class of estimators under consideration to the class of linear estimators, i.e.

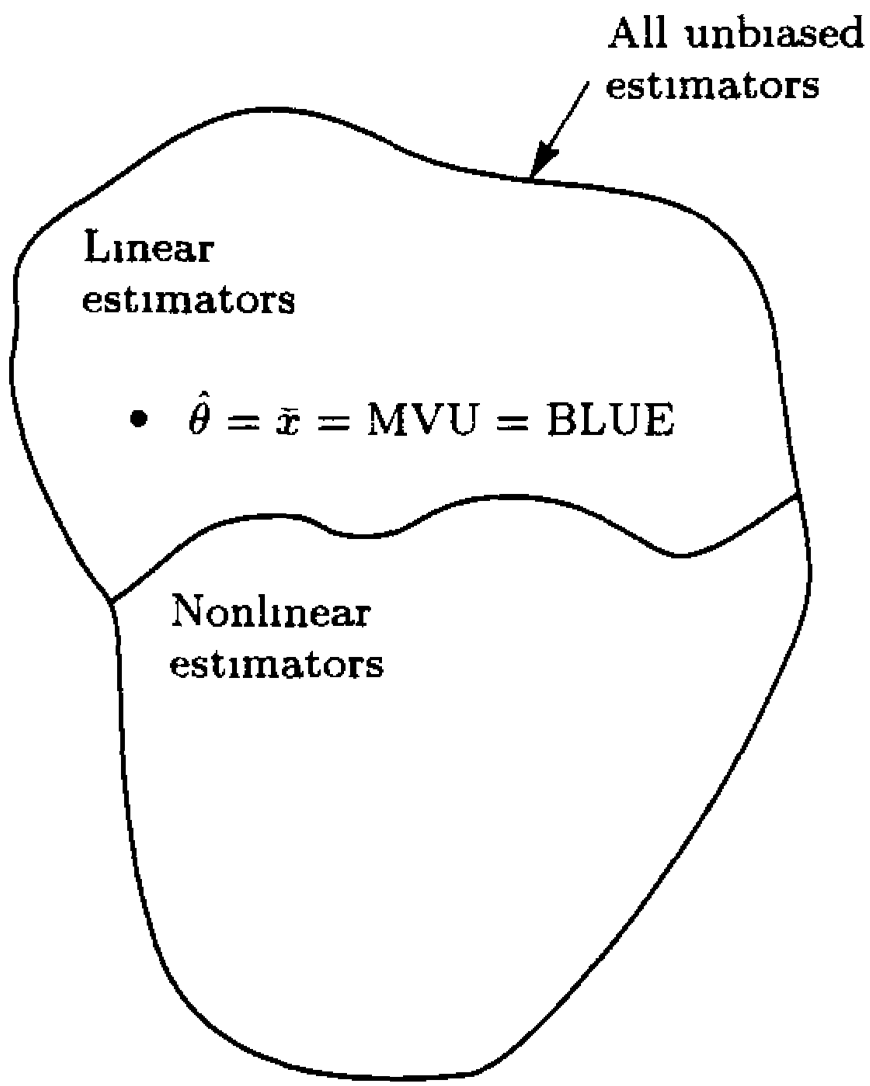
$$\hat{\theta} = \sum_{n=0}^{N-1} a_n x[n] = \mathbf{a}^T \mathbf{x}$$

- The vector a is a vector of constants, and will be “found” or “designed” or to meet certain criteria.

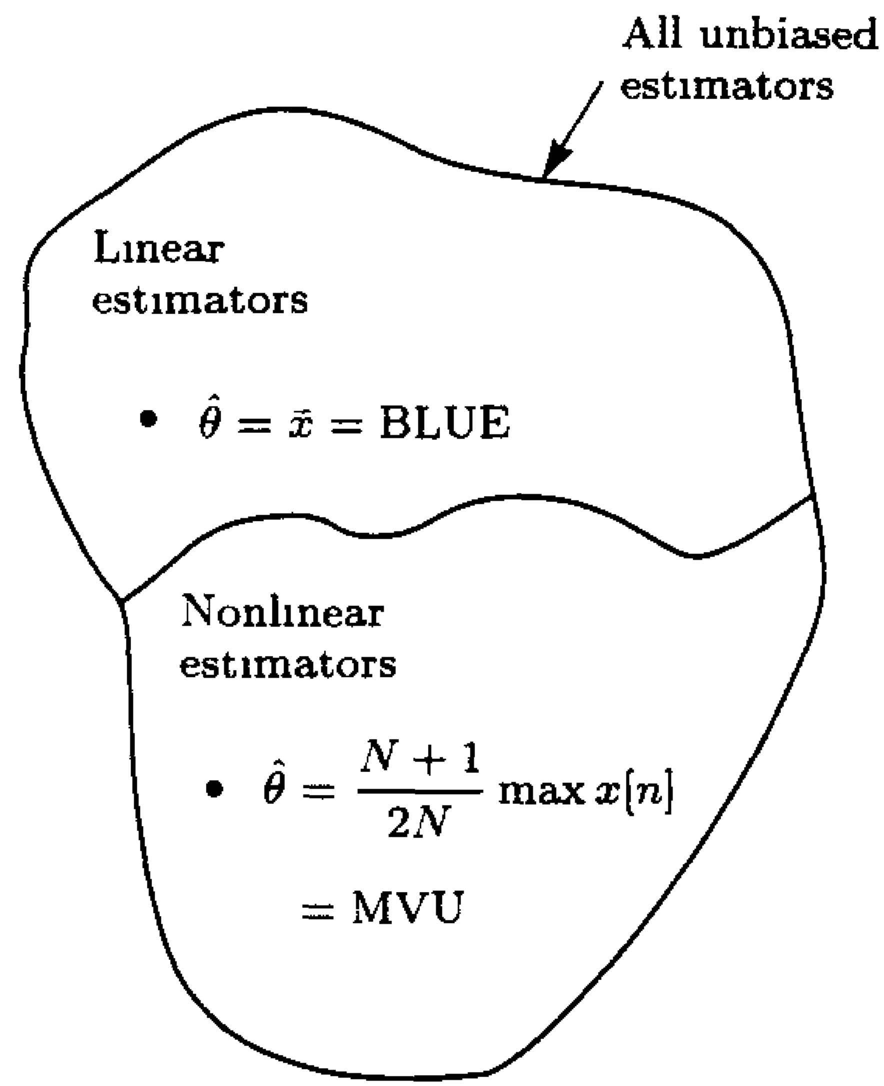
- Note that there is no reason to believe that a linear estimator will produce either an efficient estimator (meeting the CRLB), an MVUE. We are trading optimality for practicality!

--However, we can look for the estimator which is “best” in the set of linear estimators.

Best Linear Unbiased Estimator (BLUE)



(a) DC level in WGN; BLUE is optimal



(b) Mean of uniform noise; BLUE is suboptimal

Optimality of BLUE

Finding the Blue

The unbiased constraint

$$E(\hat{\theta}) = \sum_{n=0}^{N-1} a_n E(x[n]) = \theta$$

assume some form for $E(x[n])$

$E(x[n])$ must be linear in θ or

$$E(x[n]) = s[n]\theta$$

where the $s[n]$'s are *known*.

- Why?

Finding the Blue

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where the $s[n]$'s are *known*.

- Why?

Because being unbiased should hold for all θ.

Finding the Blue

$$\sum_{n=0}^{N-1} a_n E(x[n]) = \theta$$

$$\sum_{n=0}^{N-1} a_n s[n]\theta = \theta$$

$$\sum_{n=0}^{N-1} a_n s[n] = 1$$

$$\mathbf{a} = [a_0 \, a_1 \ldots a_{N-1}]^T$$

$$\mathbf{s} = [s[0] \, s[1] \ldots s[N-1]]^T$$

$$\mathbf{a}^T \mathbf{s} = 1$$

Finding the Blue

The variance

$$\begin{aligned}
\operatorname{var}(\hat{\theta}) = E & \left[\left(\sum_{n=0}^{N-1} a_n x[n] - E\left(\sum_{n=0}^{N-1} a_n x[n]\right)\right)^2\right] \\
= & \; E\left[\left(\mathbf{a}^T\mathbf{x} - \mathbf{a}^T E(\mathbf{x})\right)^2\right] \\
= & \; E\left[\left(\mathbf{a}^T(\mathbf{x} - E(\mathbf{x}))\right)^2\right] \\
= & \; E\left[\mathbf{a}^T(\mathbf{x} - E(\mathbf{x}))(\mathbf{x} - E(\mathbf{x}))^T\mathbf{a}\right] \\
= & \; \mathbf{a}^T\mathbf{C}\mathbf{a}.
\end{aligned}$$

letting

$$\mathbf{a} = [a_0\, a_1 \ldots a_{N-1}]^T$$

Finding the Blue

To find the BLUE we need to minimize the variance

$$\text{var}(\hat{\theta}) = \mathbf{a}^T \mathbf{C} \mathbf{a}$$

subject to the unbiased constraint,

$$\mathbf{a}^T \mathbf{s} = 1$$

$$\mathbf{a}_{\text{opt}} = \frac{\mathbf{C}^{-1}\mathbf{s}}{\mathbf{s}^T \mathbf{C}^{-1} \mathbf{s}}$$

$$\hat{\theta} = \frac{\mathbf{s}^T \mathbf{C}^{-1} \mathbf{x}}{\mathbf{s}^T \mathbf{C}^{-1} \mathbf{s}}$$

$$\text{var}(\hat{\theta}) = \frac{1}{\mathbf{s}^T \mathbf{C}^{-1} \mathbf{s}}$$

(Very famous, e.g. look at Capon Beamforming)

Finding the Blue

We only assume the following:

- $x[n] = s[n]\theta + w[n], \quad n = 0, 1, \cdots N-1$ for $s[n]$ known. This ensures we can find an unbiased estimator! $\quad \mathbf{x} = \mathbf{s}\theta + \mathbf{w}$
- $\mathbf{C} = E[(\mathbf{x} - E[\mathbf{x}])(\mathbf{x} - E[\mathbf{x}])^T]$ is known

Thus, we need only the first and second order moments of $\mathbf{x}$ and not the whole pdf!

"best" linear unbiased estimator

⇓

means *minimum variance.*

Finding the Blue Examples

- $x[n] = A + w[n]$, $w[n]$ not Gaussian, but independent, identically distributed of zero mean and variance σ^2
- $x[n] = A + w[n]$, $w[n]$ not Gaussian, but independent, zero mean and variance σ_n^2

Finding the Blue
Vector version

- Gauss-Markov theorem for BLUEs:

If the data are of the general linear model form

$$\mathbf{x} = \mathbf{H}\theta + \mathbf{w}$$

where $\mathbf{H}$ is a known $N \times p$ matrix, θ is a $p \times 1$ vector of parameters to be estimated and $\mathbf{w}$ is a $N \times 1$ noise vector with zero mean and covariance $\mathbf{C}$ (the PDF of $\mathbf{w}$ is otherwise arbitrary), then the BLUE of θ is

$$\hat{\theta} = \left(\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{C}^{-1}\mathbf{x}$$

and the minimum variance of θ_i is

$$var(\theta_i) = \left[(\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H})^{-1}\right]_{ii}.$$

In addition, the covariance matrix of $\hat{\theta}$ is

$$\mathbf{C}_{\hat{\theta}} = \left(\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H}\right)^{-1}.$$