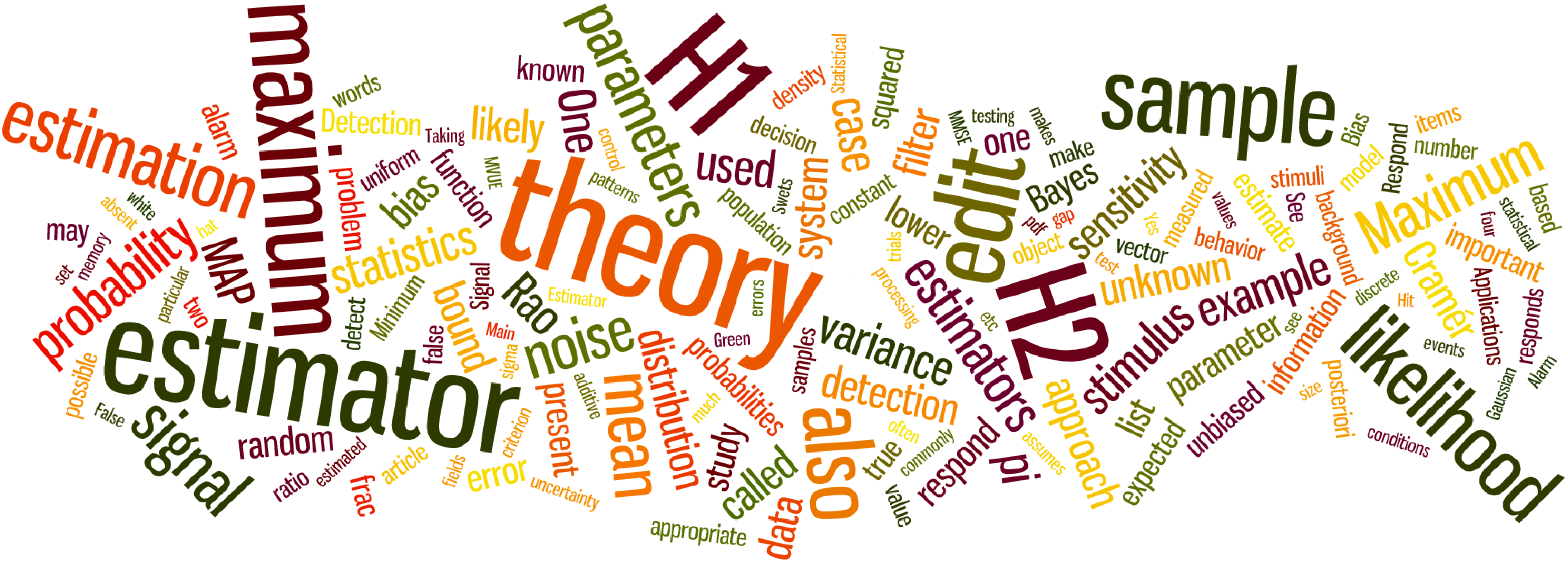

Detection and Estimation Theory
Lecture 4

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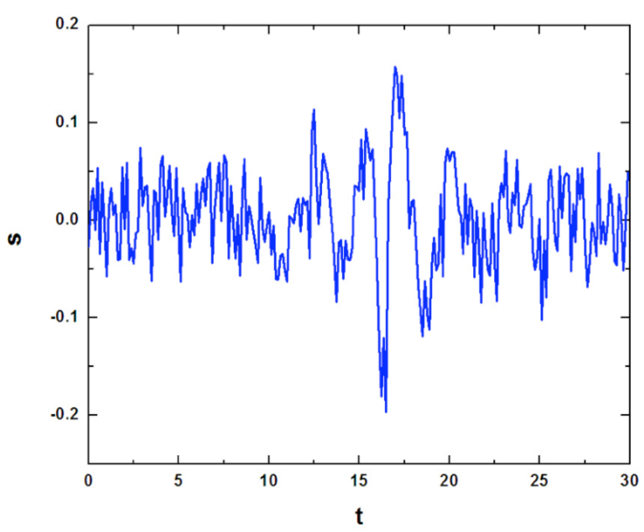
Based on ECE 531 Slides- 2011 (Prof. Natasha Devroye)

Estimation: a first example

- Estimate the DC level, A, of a signal given noisy measurements x[0], x[1], ... x[N-1] where

$$x[n] = A + w[n], \quad n = 0, 1, 2, \cdots N-1$$



x[n] are samples of this!

- Find a few estimators
- Compare their performance
  - mean?
  - variance?
  - pdf?

Estimation: a first example

- Estimators of the DC level, A

$$x[n] = A + w[n], \quad n = 0, 1, 2, \cdots N-1$$

- $\hat{A} = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$
- $\check{A} = x[0]$
- $\dot{A} = \frac{1}{N+2} \left(2x[0] + \sum_{n=1}^{N-2} x(n) + 2x[N-1] \right)$
- $\bar{A} = \frac{1}{2N} \sum_{n=0}^{N-1} x[n]$

Estimation: definitions

- <u>Parameter:</u> we wish to estimate the parameter θ from the observation(s) $\mathbf{x}$. These can be vectors $\theta = [\theta_1, \theta_2, \cdots, \theta_p]^T$ and $\mathbf{x} = [x[0], x[1], \cdots x[N-1]]^T$ or scalars.

- <u>Estimator:</u> a rule that assigns a value $\hat{\theta}$ to θ for each realization of $\mathbf{x}$.

- <u>Estimate:</u> the value of θ obtained for a given realization of $\mathbf{x}$. $\hat{\theta}$ will be used for the estimate, while θ will represent the true value of the unknown parameter.

Estimation: definitions

- Bias: an estimator $\hat{\theta}$ is called unbiased if $E(\hat{\theta}) = \theta$ for all possible θ. If this is not the case, we call $b(\theta) = E[\hat{\theta}] - \theta$ the bias. Expectation is taken with respect to $\mathbf{x}$ (or $p(\mathbf{x};\theta)$).

- Variance: the variance of an estimator $\hat{\theta}$ is defined as $var(\hat{\theta}) = E[(\hat{\theta} - E[\hat{\theta}])^2]$. Expectations are taken over $\mathbf{x}$ (meaning $\hat{\theta}$ is random, not θ.)

Vector versions....

How would you pick a ``good" estimator?

Returning to the first example…

$$x[n] = A + w[n] \qquad n = 0, 1, \ldots, N-1$$

$$\hat{A} = \frac{1}{N}\sum_{n=0}^{N-1} x[n] \longrightarrow$$

$$\begin{aligned} E(\hat{A}) &= E\left[\frac{1}{N}\sum_{n=0}^{N-1} x[n]\right] \\ &= \frac{1}{N}\sum_{n=0}^{N-1} E(x[n]) \\ &= \frac{1}{N}\sum_{n=0}^{N-1} A \\ &= A \end{aligned}$$

Unbiased. What about the other estimators?
Also what about their variance?

Combining estimators

It sometimes occurs that multiple estimates of the same parameter are available, i.e., $\{\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_n\}$.

A reasonable procedure is to combine these estimates into, hopefully,

a better one by averaging them to form

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_i.$$

Assuming the estimators are unbiased, with the same variance, and uncorrelated with each other,

$$E(\hat{\theta}) = \theta$$

and

$$\begin{aligned} \mathrm{var}(\hat{\theta}) &= \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{var}(\hat{\theta}_i) \\ &= \frac{\mathrm{var}(\hat{\theta}_1)}{n} \end{aligned}$$

Combining estimators

$$\hat{A} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$$

It is in fact combining several estimators!

Minimum variance unbiased estimation

Suppose we want to minimize the mean squared error of our estimate, this can be shown to be:

$$\begin{aligned} \text{MSE}(\hat{\theta}) &= E[(\hat{\theta} - \theta)^2] \\ &= var(\hat{\theta}) + b^2(\theta) \end{aligned}$$

Prove this!
Implications?

MVUE: the unbiased estimator $\hat{\theta}$ of the parameter θ that minimizes the estimation variance.

Minimum variance unbiased estimation

What about minimizing the MSE including bias?

Example: modified estimator

$$\check{A} = a\frac{1}{N}\sum_{n=0}^{N-1} x[n]$$

$$E(\check{A}) = aA \text{ and } \operatorname{var}(\check{A}) = a^2\sigma^2/N \longrightarrow \operatorname{mse}(\check{A}) = \frac{a^2\sigma^2}{N} + (a-1)^2A^2.$$

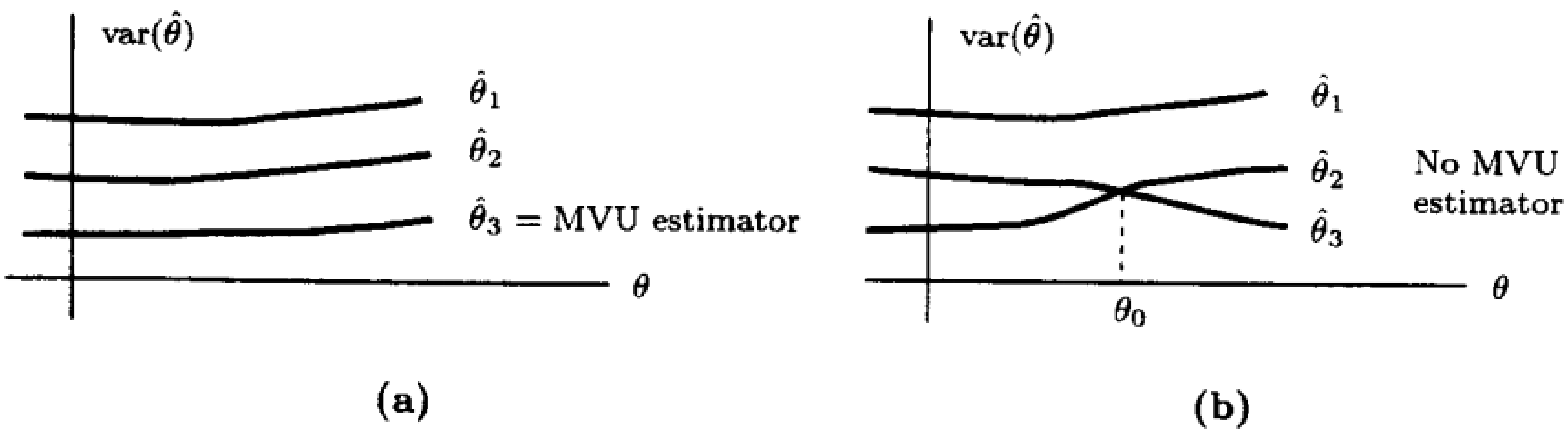
$$\frac{d\,\operatorname{mse}(\check{A})}{da} = \frac{2a\sigma^2}{N} + 2(a-1)A^2$$

$$\downarrow$$

$$a_{\mathrm{opt}} = \frac{A^2}{A^2 + \sigma^2/N}.$$

Minimum variance unbiased estimation

- Does a MVUE always exist?



Look at: **Example 2.3 - Counterexample to Existence of MVU Estimator**

Minimum variance unbiased estimation

- If it does, can we always find it?

Even if a MVU estimator exists, we may not be able to find it. There is no known "turn-the-crank" procedure which will always produce the estimator.

- Is there anything at all we can say about MVUE?

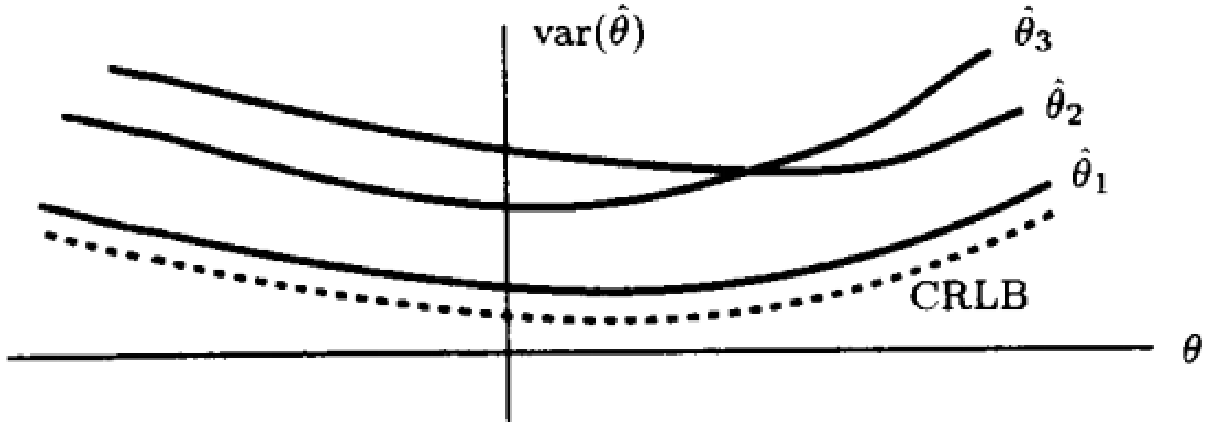


Figure 2.5 Cramer-Rao lower bound on variance of unbiased estimator

The Cramer-Rao Lower Bound

- the CRLB give a lower bound on the variance of ANY UNBIASED estimator
- does NOT guarantee bound can be obtained
- IF find an estimator whose variance = CRLB then it’s MVUE
- otherwise can use Ch.5 tools (Rao-Blackwell-Lehmann-Scheffe Theorem and Neyman-Fisher Factorization Theorem) to construct a better estimator from any unbiased one - possibly the MVUE if conditions are met

The Cramer-Rao Lower Bound

- Next lecture