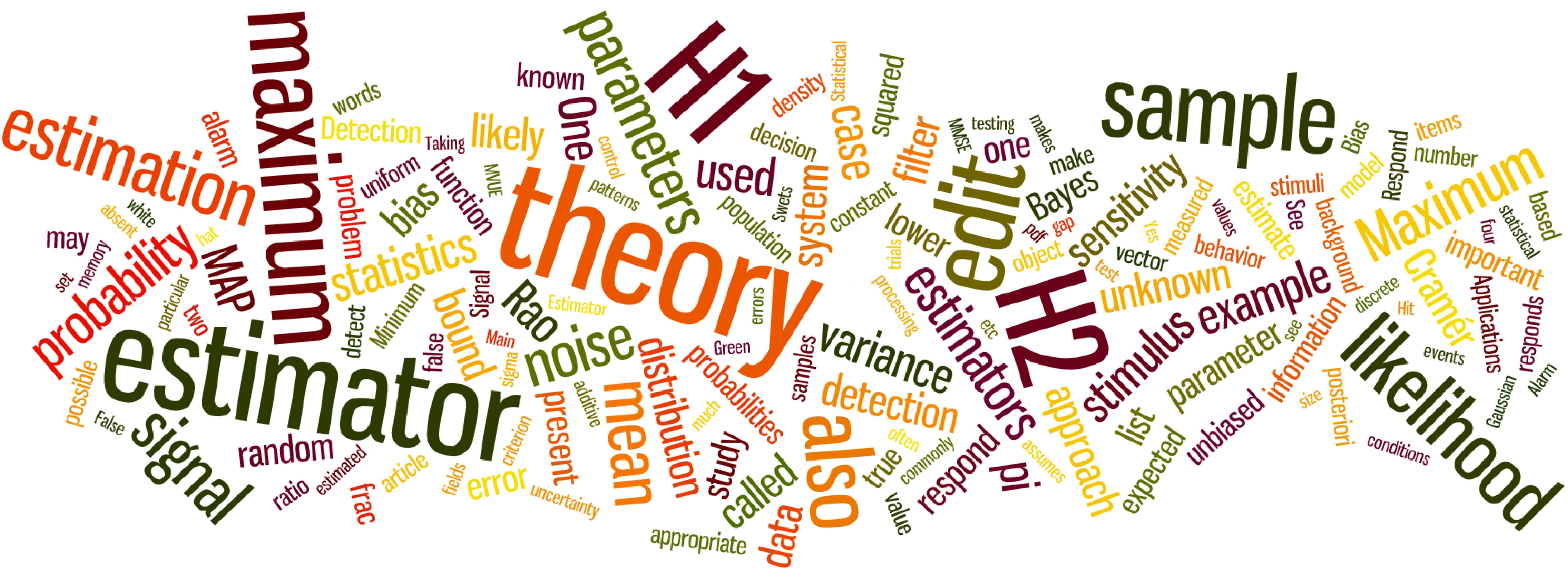

Detection and Estimation Theory
Lectures 18

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Based on ECE 531 Slides- 2011 (Prof. Natasha Devroye)

Detection Theory

- For -

Deterministic Signals

Known Signal in Gaussian Noise
>>> Matched Filter

We consider detecting the presence of a *known* signal $s[n]$, $n = 0, 1, \cdots, N-1$ in *Gaussian* noise. This means, the received signal $x[n]$, for $n = 0, 1, \cdots N-1$, is

$$\mathcal{H}_0 : x[n] = w[n]$$
$$\mathcal{H}_1 : x[n] = s[n] + w[n],$$

where $w[n]$ is for now assumed to be white with variance σ^2.

Recall that this means its autocorrelation function $r_{ww}[k] = E(w[n]w[n+k]) = \sigma^2\delta[k]$, where $\delta[k] = 1$ for $k = 0$ and 0 otherwise.

Starting from the likelihood ratio test, you can simplify the test to deciding $\mathcal{H}_1$ if the *test statistic* $T(\mathbf{x})$ is above a threshold (threshold determined by P_{FA} in Neyman-Pearson detection and by the priors and costs in Bayesian detection),

$$T(\mathbf{x}) = \sum_{n=0}^{N-1} x[n]s[n] > \gamma'$$

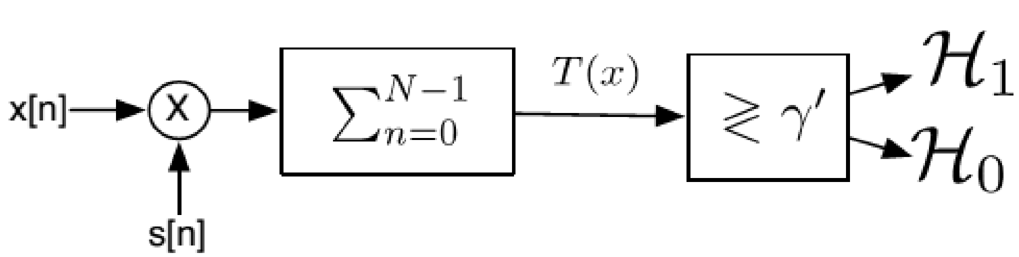
>>> Matched Filter
a.k.a. Correlator, or Replica-Correlator

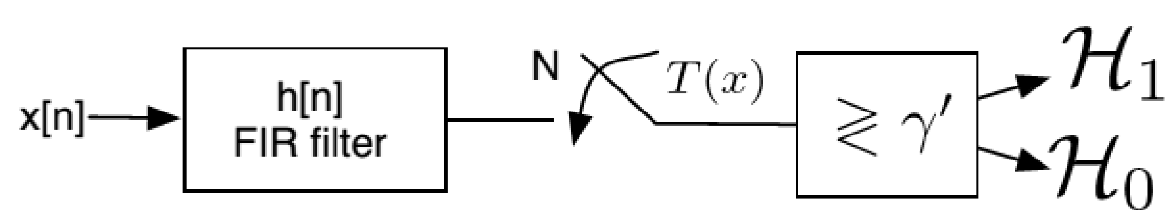
We consider detecting the presence of a *known* signal $s[n]$, $n = 0, 1, \cdots, N-1$ in *Gaussian* noise. This means, the received signal $x[n]$, for $n = 0, 1, \cdots N-1$.

$$T(\mathbf{x}) = \sum_{n=0}^{N-1} x[n]s[n] > \gamma'$$

This test is called a *correlator* or *replica-correlator*. It is optimal in white Gaussian noise.

Another equivalent but more "signal processing" type approach to detecting $s[n]$ in the received $x[n]$ is to use a so-called *matched filter*. Here we view send the received signal $x[n]$ through a linear time invariant filter with a finite impulse response (FIR) $h[n] = s[N-1-n]$ for $n = 0, 1, \cdots N-1$. This impulse response is "matched" to the signal, it's a flipped version of it. We make our decision by sampling the output of the filter at time $N-1$ and comparing it with the threshold γ', as before.





Detection Theory

- For -

Random Signals

Motivation & Formulation

- Some processes are better represented as random (e.g. speech)
- rather than assume completely random, assume signal comes from a random process of known **covariance structure**

Consider a binary hypothesis testing model of the following form:

$$\mathcal{H}_0 : x[n] = w[n]$$
$$\mathcal{H}_1 : x[n] = s[n] + w[n],$$

where $\mathbf{w} \sim \mathcal{N}(0, \mathbf{C_s})$ and $\mathbf{s} \sim \mathcal{N}(\mu_\mathbf{s}, \mathbf{C_s})$ and $\mathbf{s}, \mathbf{w}$ are independent. We have $n = 0, 1, \cdots N-1$ (N samples).

Detection for Random Signals

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We thus can discriminate between the two hypothesis based on both their means and covariances. Taking the likelihood ratio and simplifying, our test statistic $T(\mathbf{x})$ can be shown to be:

$$T(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T \left[\mathbf{C_w}^{-1}\mathbf{C_s}(\mathbf{C_s} + \mathbf{C_w})^{-1}\right]\mathbf{x} + \mathbf{x}^T(\mathbf{C_s} + \mathbf{C_w})^{-1}\mu_\mathbf{s}$$

The test statistic has a quadratic term in $\mathbf{x}$ (intuitively account for the different variances) as well as a linear term in $\mathbf{x}$ accounting for the different means.

Detection for Random Signals
>>> Various Scenarios

Energy detectors? Suppose we have WGN of variance σ^2 and a signal which is a zero-mean Wide Sense Stationary Gaussian process with variance σ_s^2?

Then $\mathbf{C_s} = \sigma_s^2\mathbf{I}, \mu_\mathbf{s} = \mathbf{0}$ and $\mathbf{C_w} = \sigma^2\mathbf{I}$. Then the test statistic becomes $T(\mathbf{x}) = \sum_{n=0}^{N-1} x^2[n]$ which is then compared to a threshold. This is just an energy detector, which makes sense as the only difference between the signal and the noise is its variance.

Detection for Random Signals
>>> Various Scenarios

Estimator-correlator? Suppose we have WGN of variance σ^2 and a signal of zero mean and covariance $\mathbf{C_s}$. Then the test statistic becomes $T(\mathbf{x}) = \sigma^2\mathbf{x}^T\left[\mathbf{C_s}(\mathbf{C_s}+\sigma^2\mathbf{I})^{-1})\mathbf{x}\right]$, which may be re-written as a new test statistic $T'(\mathbf{x} = \mathbf{x}^T\hat{\mathbf{s}}$ for $\hat{\mathbf{s}} = \mathbf{C_s}(\mathbf{C_s}+\sigma^2\mathbf{I})^{-1})\mathbf{x}$.

Interestingly, $\hat{\mathbf{s}}$ is the Minimum Mean Squared Error Estimate of the signal $\mathbf{s}$ given the received data $\mathbf{x}$ (we will see this later). So what we are in essence doing is correlating the received signal with an *estimate* of the signal $\mathbf{s}$, hence the name *estimator-correlator.*

Detection for Random Signals
>>> Various Scenarios

Estimator-correlator with colored noise? We now have $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{C_w})$ and $\mathbf{s} \sim \mathcal{N}(\mathbf{0}, \mathbf{C_s})$. The test statistic becomes

$$T(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T \mathbf{C_w}^{-1}\left[\mathbf{C_s}(\mathbf{C_s} + \mathbf{C_w})^{-1}\,\mathbf{x}\right] = \frac{1}{2}\mathbf{x}^T \mathbf{C_w}^{-1}\hat{\mathbf{s}}$$

This looks like the generalized matched filter (matched filter in colored noise), where $\hat{\mathbf{s}}$ is now an estimate of the signal given by $\hat{\mathbf{s}} = \mathbf{C_s}(\mathbf{C_s} + \mathbf{C_w})^{-1}\,\mathbf{x}$ rather than the known signal we had before.