Detection and Estimation Theory
Lectures 18

Mojtaba Soltanalian- UIC
msol@uic.edu
http://msol.people.uic.edu

Based on ECE 531 Slides- 2011 (Prof. Natasha Devroye)
Detection Theory

- For -

Deterministic Signals
Known Signal in Gaussian Noise

>>> Matched Filter

We consider detecting the presence of a known signal \( s[n], n = 0, 1, \cdots, N - 1 \) in Gaussian noise. This means, the received signal \( x[n] \), for \( n = 0, 1, \cdots N - 1 \), is

\[
\begin{align*}
\mathcal{H}_0 &: x[n] = w[n] \\
\mathcal{H}_1 &: x[n] = s[n] + w[n],
\end{align*}
\]

where \( w[n] \) is for now assumed to be white with variance \( \sigma^2 \).

Recall that this means its autocorrelation function \( r_{ww}[k] = E(w[n]w[n+k]) = \sigma^2 \delta[k] \), where \( \delta[k] = 1 \) for \( k = 0 \) and 0 otherwise.

Starting from the likelihood ratio test, you can simplify the test to deciding \( \mathcal{H}_1 \) if the test statistic \( T(x) \) is above a threshold (threshold determined by \( P_{FA} \) in Neyman-Pearson detection and by the priors and costs in Bayesian detection),

\[
T(x) = \sum_{n=0}^{N-1} x[n]s[n] > \gamma'
\]
Matched Filter
a.k.a. Correlator, or Replica-Correlator

We consider detecting the presence of a *known* signal $s[n]$, $n = 0, 1, \cdots, N - 1$ in *Gaussian* noise. This means, the received signal $x[n]$, for $n = 0, 1, \cdots N - 1$.

$$T(x) = \sum_{n=0}^{N-1} x[n]s[n] > \gamma'$$

This test is called a *correlator* or *replica-correlator*. It is optimal in white Gaussian noise.

Another equivalent but more “signal processing” type approach to detecting $s[n]$ in the received $x[n]$ is to use a so-called *matched filter*. Here we view send the received signal $x[n]$ through a linear time invariant filter with a finite impulse response (FIR) $h[n] = s[N - 1 - n]$ for $n = 0, 1, \cdots N - 1$. This impulse response is “matched” to the signal, it’s a flipped version of it. We make our decision by sampling the output of the filter at time $N - 1$ and comparing it with the threshold $\gamma'$, as before.
Detection Theory

- For -

Random Signals
Motivation & Formulation

- Some processes are better represented as random (e.g. speech)

- rather than assume completely random, assume signal comes from a random process of known \textbf{covariance structure}

Consider a binary hypothesis testing model of the following form:

\[ \mathcal{H}_0 : x[n] = w[n] \]
\[ \mathcal{H}_1 : x[n] = s[n] + w[n], \]

where \( w \sim \mathcal{N}(0, C_s) \) and \( s \sim \mathcal{N}(\mu_s, C_s) \) and \( s, w \) are independent. We have \( n = 0, 1, \cdots N - 1 \) (\( N \) samples).
Detection for Random Signals

Consider a binary hypothesis testing model of the following form:

$$
\mathcal{H}_0 : x[n] = w[n] \\
\mathcal{H}_1 : x[n] = s[n] + w[n],
$$

where $w \sim \mathcal{N}(0, C_w)$ and $s \sim \mathcal{N}(\mu_s, C_s)$ and $s, w$ are independent. We have $n = 0, 1, \cdots N - 1$ ($N$ samples).

We thus can discriminate between the two hypothesis based on both their means and covariances. Taking the likelihood ratio and simplifying, our test statistic $T(x)$ can be shown to be:

$$
T(x) = \frac{1}{2} x^T \left[ C_{w}^{-1} C_s (C_s + C_{w})^{-1} \right] x + x^T (C_s + C_{w})^{-1} \mu_s
$$

The test statistic has a quadratic term in $x$ (intuitively account for the different variances) as well as a linear term in $x$ accounting for the different means.
Detection for Random Signals

>>> Various Scenarios

*Energy detectors?* Suppose we have WGN of variance $\sigma^2$ and a signal which is a zero-mean Wide Sense Stationary Gaussian process with variance $\sigma_s^2$?

Then $C_s = \sigma_s^2 I$, $\mu_s = 0$ and $C_w = \sigma^2 I$. Then the test statistic becomes $T(x) = \sum_{n=0}^{N-1} x^2[n]$ which is then compared to a threshold. This is just an energy detector, which makes sense as the only difference between the signal and the noise is its variance.
**Detection for Random Signals**

**>>> Various Scenarios**

*Estimator-correlator?* Suppose we have WGN of variance \( \sigma^2 \) and a signal of zero mean and covariance \( C_s \). Then the test statistic becomes \( T(x) = \sigma^2 x^T [C_s (C_s + \sigma^2 I)^{-1}] x \), which may be re-written as a new test statistic \( T'(x = x^T \hat{s} \text{ for } \hat{s} = C_s (C_s + \sigma^2 I)^{-1}) x \).

Interestingly, \( \hat{s} \) is the Minimum Mean Squared Error Estimate of the signal \( s \) given the received data \( x \) (we will see this later). So what we are in essence doing is correlating the received signal with an *estimate* of the signal \( s \), hence the name *estimator-correlator*. 
Detection for Random Signals

>>> Various Scenarios

*Estimator-correlator with colored noise?* We now have \( w \sim N(0, C_w) \) and \( s \sim N(0, C_s) \). The test statistic becomes

\[
T(x) = \frac{1}{2} x^T C_w^{-1} \left[ C_s (C_s + C_w)^{-1} x \right] = \frac{1}{2} x^T C_w^{-1} \hat{s}
\]

This looks like the generalized matched filter (matched filter in colored noise), where \( \hat{s} \) is now an estimate of the signal given by \( \hat{s} = C_s (C_s + C_w) x \) rather than the known signal we had before.