ECE 531 Detection and Estimation Theory
Midterm 1
February 16, 2016. 2:00-3:15 in LH312.

- Many of the problems have short to-the-point answers. Use your time wisely!
- Using the course textbook and slides are permitted during the exam.
- Make sure you explain your answers in a way that illustrates your understanding of the problem. Ideas are important, not just the calculation.
- Partial marks will be given.
- Write all answers directly on this booklet.

Your name: 

Your UIN: 

Intentional use or attempt to use unauthorized assistance including material, information, or people in this exam will lead to penalties such as a failing grade. College of Engineering and University guidelines will be followed.

I hereby certify that all the answers given here are fully mine, and no unauthorized assistance is used.

Your signature: 

The exam has 4 questions, for a total of 20 points.

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1. minimal error vs minimal variance estimation.
   (a) (2 points) Prove that
   \[ MSE = Var + Bias^2. \] \hfill (1)
   
   (b) (3 points) The MVUE is defined as the estimator that has the minimal variance among all unbiased estimators. In light of (1), discuss the validity of the following argument:

   “We can have an estimator with a constant bias—a bias whose value is \textit{a priori} known. By injecting such a bias, we can get a lower variance of estimation, even lower than that of the MVUE counterpart.”

Points earned: ___________ out of a possible 5 points
2. *Centralized estimation in sensor networks.* Assume $N$ sensors observing a given deterministic parameter $\mu$. The sensors have independent zero-mean AWGN with different variances given by

$$\sigma_1^2, \sigma_2^2, \ldots, \sigma_N^2.$$  

(2)

The observed values from the sensors $\{\hat{\mu}_n\}_{n=1}^N$ are transmitted to a fusion center so that the parameter $\mu$ can be estimated in a centralized manner.

(a) (3 points) Propose an optimal linear combination strategy for the fusion center to obtain an unbiased estimate of $\mu$ with minimal variance. To accomplish this task, you should determine $\{w_n\}_{n=1}^N$ in the following estimator:

$$\hat{\mu} = \sum_{n=1}^N w_n \hat{\mu}_n.$$  

(3)

(b) (2 points) Describe the minimal achievable variance in terms of measurement variances of the sensors. *Hint:* The answer should not be parametrized by $\{w_n\}_{n=1}^N$. 

Points earned: ____________ out of a possible 5 points
3. Centralized estimation in sensor networks (the channel-aware case). We assume that \( N \) single-antenna sensors in a distributed sensor network independently observe an unknown but deterministic complex-valued parameter \( \mu \), according to the following model for sensor \( i \):

\[
y_i = \mu + v_i,
\]

where \( v_i \) is complex-valued Gaussian observation noise with variance \( \sigma_i^2 \). The noise is assumed to be independent from sensor to sensor. The \( i^{th} \) sensor transmits the signal \( y_i \) to the fusion center (FC). Assuming a FC with \( M \) antennas, the vector signal received at the FC can be expressed as

\[
y = \mu \mathbf{H} \mathbf{1} + H \mathbf{v} + \mathbf{n}
\]

where \( \mathbf{H} = [h_1 \ h_2 \ \cdots \ h_N] \) with \( h_i \in \mathbb{R}^M \) being the channel vector between the \( i^{th} \) sensor and the FC, \( \mathbf{1} \) is the all-one vector of length \( N \), \( \mathbf{v} \) is the sensor measurement noise vector with covariance \( \mathbf{V} = \mathbb{E}\{\mathbf{v}\mathbf{v}^T\} = \text{Diag}(\sigma_1^2, \sigma_2^2, \cdots, \sigma_N^2) \), and \( \mathbf{n} \) is complex Gaussian noise at the FC with covariance \( \mathbb{E}\{\mathbf{n}\mathbf{n}^T\} = \sigma_n^2 \mathbf{I}_M \), where \( \mathbf{I}_M \) is the identity matrix of \( \mathbb{R}^M \).

Assuming that the FC is aware of the channel matrix \( \mathbf{H} \), the noise covariance \( \mathbf{V} \) and \( \sigma_n^2 \):

(a) (3 points) What is the MVUE expression for estimating \( \mu \) at the FC?

(b) (2 points) Determine the estimation variance at the FC if the MVUE is used. Is the CRLB tight in this problem?

Points earned: ___________ out of a possible 5 points
4. Distributed estimation in sensor networks and consensus. Herein, we assume that a connected network of sensors handles the task of estimating the deterministic parameter $\mu$ without using a fusion center. The estimation scheme comprises of two stages, namely (i) measurement, in which every sensor measures (and forms an unbiased estimate of) the unknown parameter, and (ii) consensus, which can be explained as follows: The process will be performed step-by-step through time. At each step, each sensor node forms an unbiased estimate of $\mu$ by calculating a weighted average (such as in Problem 2) of estimates collected from the nodes connected to it, including its own current estimate—see the illustrative figure below. Then, the obtained estimate will be used to update the current estimate of the sensor node, and the process continues. Note that no new measurement is taking place during the consensus stage.

(a) (2 points) Show that a consensus will be reached, i.e. the estimates at all sensor nodes converge to the same value $\hat{\mu}$, which is an unbiased estimate of $\mu$.

(b) (3 points) Given the network connectivity and sensor information at the measurement stage, propose a mathematical approach in order to analytically calculate $\hat{\mu}$.

Points earned: __________ out of a possible 5 points