This paper is an attempt to present a short, unified discussion of the radar detection, parameter estimation, and multiple-signal resolution problems--mostly from a philosophical rather than a detailed mathematical point of view. The purpose is, hopefully, to make it possible in at least some limited sense to reason back from appropriate measures of desired radar performance to specifications of the necessary values of the related radar parameters. Specifically four measures of performance quality are considered:

1. The reliability of detection,
2. The accuracy with which target parameters can be estimated,
3. The extent to which such estimates can be made without ambiguity,
4. The degree to which two or more different target echoes can be separated or resolved.

It is argued that the radar synthesis problem can be split into two more-or-less independent phases. First, adjust such parameters as those appearing in the radar equation so that the received signal energy is sufficiently large for the degree of reliability of detection desired. The required value of energy is almost entirely independent of the character of the received echo waveform. The second phase is, then, to select the waveform in such a way that accuracy, ambiguity, and resolution requirements are met. The limitations on what can be achieved in terms of these three quality measures are discussed in relation to an uncertainty principle. For purposes of illustration several novel waveforms having unusual and useful properties are described.

Most radar design engineers today are acquainted with at least the rudiments of statistical methods and probability concepts. They have studied aspects of detection theory and mastered the operational methods of signal and system analysis. They speak knowingly of "matched filters" and "uncertainty principles," but often this knowledge is fragmentary--quite useful for radar analysis, but fundamentally inadequate for radar synthesis. What is often missing is a sense of perspective, an appreciation of the relative importance of and the interconnections between isolated bits of knowledge--in a phrase, a radar detection philosophy. The rather ambitious purpose of this paper is to attempt to state such a philosophy--or better a part of such a philosophy since we shall ignore many aspects of the radar detection problem. Specifically we shall not even mention many practical questions such as implementation, effect of system instabilities, approximations, and distortions, countermeasures, etc. Moreover, we shall for various reasons to be discussed consider only "search" type radar applications.

Our intention, thus, is to discuss a theory. Now, the ultimate purpose of any theory in applied science is always to achieve some type of synthesis, i.e., to make it possible to reason back from effects to causes, or from desired performance to system parameters. To be successful, then, our theory must meet three conditions:

1. The model on which the theory is based must at least approximately represent the actual physical situation;
2. The theory must yield a fundamental, complete, and consistent set of parameters and concepts in terms of which both the desired performance and the radar system can be uniquely specified;
3. The theory must include all upper bounds, limiting relationships, or realizability conditions, which prevent the simultaneous achievement of an arbitrary set of parameters.

These three points constitute a rough outline of this paper. More specifically we shall first postulate a model of both the target situation and the radar. We shall then consider two very restricted special cases corresponding to a single target with discrete parameter distributions. Despite the restrictions, a discussion of these cases will lead to quite general statements about those parameters and limiting conditions which relate to the general question of reliability of detection. Next we shall pass to the case of continuous parameter distributions and consider the questions of accuracy and ambiguity and their relation to what might be called the radar Uncertainty Principle. This section will be illustrated by a number of examples, some of which are rather novel. Finally we shall consider briefly and qualitatively the case of multiple targets and the question of resolution.

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We shall begin with the postulation of a model. For detection purposes we are principally interested in the received waveform and the way in which it is related to the parameters of the targets and the transmitted waveform. We can avoid a number of trivial steps in the argument if we choose our postulates so as to define directly the received waveform. Hence, we shall assume that:

1. The volume examined by the radar contains a number of point scatterers whose individual properties can be completely described by:
   a. the amplitude of the return from each,
   b. the range of the scatterer or delay in the echo,
   c. the velocity of the scatterer or Doppler shift in the echo;

2. The effective duration of the echo from each scatterer is limited, known and independent of the properties of the scatterer;

3. The amplitude of the echo and the velocity of the scatterer are constants, at least during the effective echo duration.

A fourth assumption is required to specify the actual shape of the received waveform, but since this is primarily a matter of nomenclature and requires some development we shall postpone it for the moment. It is easy to raise questions about the necessity, rationality, and implications of the assumptions listed above. Nevertheless, we believe that they are the simplest set of constraints which preserve, at least in some rudimentary form, the major aspects of the radar problem. Moreover, they are the most common assumptions, implicit if not expressed, in most discussions of the radar problem, and they have in general the pragmatic justification of leading to mathematical and philosophical problems which are, at least in principle, amenable to solution. Physically, \( S(t) \) will be interpreted as the return from a "unit" fixed target at zero range. Consequently, except for the effect of antenna scanning or equivalent means of limiting the echo duration, \( S(t) \) and \( y(t) \) can be considered as the envelope and phase variation of the transmitted waveform, and hence along with \( \omega_0 \) are assumed known to the receiver. Thus, using the narrow band assumption, the echo from a target having a range delay \( r \) and a Doppler frequency shift \( \omega \) rad/sec, \( \omega_0 \) will be represented as

\[
S_w(r, \tau) = \Re \left[ S_w(t) e^{j\omega_0 t} \right]
\]

where

\[
S_w(t) = |S_w(t)| e^{jyt(t)}
\]

so that

\[
S_w(r, \tau) - |S_w(t)| e^{j\omega_0 t}
\]

The additional assumption will be made that \( |S_w(t)| \) and \( y(t) \) vary slowly compared with \( \omega_0 t \) so that the usual narrow-band assumptions will be justified. For example, it is convenient to normalize the energy level of \( S_w(t) \) by specifying that

\[
\int_0^\infty |S_w(t)|^2 dt = 1
\]

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\]

Returning now to the fourth assumption, we desire to introduce some symbolic notation for the received waveform and its relation to the transmitted waveform and the target parameters. The first three assumptions permit us to represent the echo signal received at any moment in terms of a canonical signal

\[
s(t) = \Re \left[ S_w(t) e^{j\omega_0 t} \right]
\]

where

\[
S_w(t) = |S_w(t)| e^{jyt(t)}
\]

\[
\omega_0 = \text{carrier frequency}
\]

so that

\[
s(t) - |S_w(t)| e^{j\omega_0 t}
\]

The additional assumption will be made that \( |S_w(t)| \) and \( y(t) \) vary slowly compared with \( \omega_0 t \) so that the usual narrow-band assumptions will be justified. For example, it is convenient to normalize the energy level of \( s(t) \) by specifying that

\[
\int_0^\infty |s(t)|^2 dt = 1
\]

1. Complex notation is employed for simplicity at a later stage and should not be considered in any sense fundamental or mysterious.

2. Although the limits of integration in (1-2), as in other similar integrals to follow, are given as \(-\infty\) and \(\infty\), it should be remembered that, in accordance with the second basic assumption, \( s(t) \) is assumed to be zero outside some relatively short interval of time.
Finally our fourth assumption is that the total received signal can be represented as
\[ r(t) = n(t) + \sum_{k} s_{k}(t - \tau_{k}) \]
where \( n(t) \) represents essentially white Gaussian noise of known power density, \( N_{0} \) watts/ops (double-sided spectrum).

The total received signal energy is
\[ \int_{-\infty}^{\infty} (r(t) - \bar{r})^2 dt \]
= echo signal energy.

The Posterior Probability

Given \( r(t) \), our problem roughly is to determine the number and parameters of the targets which are present. Of course we cannot expect to do this with absolute certainty because of the noise, if for no other reason. The best that we can hope to achieve is to be able to attach a probability to the truth of any proposition made about the target situation. Specifically we shall assert that a specification for each possible set of values of \( \alpha, \tau, \) and \( \omega \) of the probability that there is a target present with those parameters constitutes a complete statement of our knowledge about the target situation obtained from \( r(t) \). Considered as a function of \( \alpha, \tau, \) and \( \omega \), this is called the a posteriori probability distribution which we shall denote by \( P(\alpha, \tau, \omega) \).

However, as soon as we try to compute \( P(\alpha, \tau, \omega) \) for some particular \( r(t) \), a difficult appears. This is, of course, the fact that \( P(\alpha, \tau, \omega) \) for each particular \( \alpha, \tau, \) and \( \omega \) depends on our expectation prior to receiving \( r(t) \) that a target with that particular set of parameters would be present, i.e., \( P(\alpha, \tau, \omega) \) depends on the a priori probability distribution \( P(\alpha, \tau, \omega) \). Other things being equal, the more likely a target is before receiving \( r(t) \), the more likely it is afterwards. Specifically it can be shown that if \( n(t) \) is white Gaussian noise and if we know, for example, that there is at most one target present, then the a posteriori probability that a target is present and has the particular parameters \( \alpha, \tau, \) and \( \omega \) is

\[ P(\alpha, \tau, \omega) = \frac{1}{k} P(\alpha, \tau, \omega) e^{-\int_{-\infty}^{\infty} (r(t) - \bar{r})^2 dt / 2N_{0}} \]

where \( k \) = normalization constant, independent of \( \alpha, \tau, \) and \( \omega \), so chosen that the sum or integral of \( P(\alpha, \tau, \omega) \) over all possible values (including \( \alpha, \tau, \omega \)) of the parameters is 1.

The difficulty, of course, is that in many cases the radar designer—and indeed the radar user—have little or no knowledge about \( P(\alpha, \tau, \omega) \). To quote an example of Woodward's, what after all is the a priori probability of observing an aircraft on a given radar set at a range of ten miles at nine o'clock tomorrow morning? It is not even clear that this question has any meaning in the sense of mathematical as opposed to subjective probability, since an ensemble of like situations is hard to imagine.

It has been argued with some justification, that, from the point of view of radar design at least, the dependence of \( P(\alpha, \tau, \omega) \) on \( r(t) \) is not very important. The argument essentially depends upon two observations:

1. The only way in which the particular signal received adds to our knowledge of any attribute of the situation is through the integral

\[ P(\alpha, \tau, \omega) = \frac{1}{k} P(\alpha, \tau, \omega) \int_{-\infty}^{\infty} e^{-\int_{-\infty}^{\infty} (r(t) - \bar{r})^2 dt / 2N_{0}} \]

Thus any receiver which computes the integral \( (2-2) \) for all possible values of \( \alpha, \tau, \) and \( \omega \) is preserving all of that information in \( r(t) \) relevant to any decisions about the presence or parameters of an echo signal. Furthermore, the output of this receiver is just sufficient in that any further operations on \( P(\alpha, \tau, \omega) \) will either destroy useful information or imply assumptions concerning the form of \( P(\alpha, \tau, \omega) \). It is in this sense at least that a receiver performing the cross-correlation type operation \( (2-2) \) may be said to be optimum—and the structure of this receiver does not depend on \( P(\alpha, \tau, \omega) \).

2. The way in which \( P(\alpha, \tau, \omega) \) enters the computation of \( P(\alpha, \tau, \omega) \) is purely as a scale factor or weighting. Thus its influence on the equipment design is essentially that of a gain control adjustment which need not greatly worry the radar designer since it can be left to the user to set this control in accordance with the situation and his own particular judgements.

At best it seems to us this argument amounts to ducking the issue. There are two reasons why we cannot get rid of the a priori probability problem so easily.

1. Contrary to the more usual practice we shall employ throughout a double-sided spectrum, i.e., including both positive and negative frequencies. Hence, for example, the noise power output of a filter with frequency response \( H(\omega) \) will be

\[ \frac{N_{0}}{2\pi} \int |H(\omega)|^2 d\omega. \]

For a receiver with noise figure \( F \), \( N_{0} \) is given by \( N_{0} = \frac{1}{2} k T \), \( k \) = Boltzmann's constant, \( T \) = absolute temperature.

[1]. Superscripts in brackets refer to the bibliography.
1. Sooner or later in every system design it is necessary to make decisions, e.g., to proceed from noncommittal probabilities to firm statements right or wrong, that targets are located there and there and there. Such decisions are the rightful concern of the radar designer, if for no other reason than that he may very well be called upon to make them. But as soon as decisions are required the a priori difficulty comes back to haunt us—mixed up now with another disturbing subject, the question of risks or value judgments. It is idle to pretend that decisions based on maximum likelihood, Neyman-Pearson, or minimax criteria avoid the difficult since the selection of such a criterion really amounts in effect to an assumption about the form of $P(R|\tau)$. 

2. The values of $P(R|\tau)$ for all possible sets of $R$, $\tau$, and $\omega$ represent quite a lot of data—more, indeed, than the customer may wish or can assimilate. Some of these parameters, e.g., $P$, may not contribute much information and the customer may very well suggest that the designer get rid of them by integrating $P(R|\tau)$ over these parameters. In this event the necessity for knowing or assuming $P(R|\tau)$ is completely unavoidable.

Throughout the remainder of this paper we shall assume that $P(R|\tau)$ is known or has been more or less arbitrarily selected. This really constitutes, of course, a fifth postulate and should perhaps be listed in the preceding section. In the last analysis the justification for this assumption lies in the fact that we are dealing with a theory. In the modern axiomatic sense of the word—a theory cannot be tested on the basis of its truth, but only for its utility. The usefulness of the present theory, including the assumption of known a priori probabilities, has, we believe, been demonstrated many times. In certain cases concerning the a priori distributions are relatively innocuous. For example, suppose again that there is known to be only one target present and that $\alpha/\beta$ is known to be reasonably large compared with 1. Then $\Lambda$, as defined in (2-2), will, as a result of the narrow-band assumption, be an almost periodic function of $\tau$ in the vicinity of the true values of $\tau$ and $\omega$, and, indeed, will be almost a sine wave of frequency $\omega$, and amplitude $\alpha/\beta$. The corresponding period, expressed in terms of range, is one-half the wave length at the frequency $\omega$, i.e., in most cases a few feet or less. Appealing to a sort of general principle of continuity, it is certainly reasonable to assume that $P(R|\tau)$ is essentially constant over variations of $\tau$ corresponding to as small a range difference as a few feet. Thus $P(R|\tau)$ considered as a function of $\tau$ can be expected to alternate rapidly between a large and a small value. Physically the implication is that the range of the target can perhaps be determined quite "accurately" (small fraction of a wave length), but that this determination is highly "ambiguous" (multiples of half-wavelengths). As a result of this ambiguity the fine structure of $P(R|\tau)$ is essentially meaningless in most cases and the logical operation to perform is to replace $P(R|\tau)$ by local sums of $P(R|\tau)$ over half-wavelength intervals, assuming ranges within this interval equally likely. Taking advantage of the narrow-band assumption, this operation is readily carried out by treating the $\omega$ term in $P(R|\tau)$ as an independent random phase angle, $\omega$, assumed to be uniformly distributed over $0<\omega$. Averaging over $\omega$ we obtain

$$P(R|\tau) = e^{-2\pi \omega} \frac{\Lambda(\tau)}{\sqrt{\lambda \pi}}$$

where

$$\Lambda = \int \frac{\Lambda(\tau)}{\sqrt{\lambda \pi}} e^{j\omega \tau} d\tau$$

and $\Lambda$ is the Bessel function of imaginary argument. $P(R|\tau)$ can be thought of as essentially the "envelope" of $P(R|\tau)$ given by (2-1). More precisely $\Lambda(\tau)$ is a monotonic function of $\tau$ so that $\Lambda$, plays the same role of a sufficient statistic with respect to $P(R|\tau)$ as $\Lambda$ plays with respect to $P(R|\tau)$. Now it is easy to show that $\Lambda$, considered as a function of $\tau$, may be interpreted physically (in the case at least when $\Lambda(\tau)$ is precisely the envelope of $\Lambda$, which is certainly an intuitively satisfying result considering the assumptions).

1. But not in all cases; e.g., not if the radar being considered is one station of an interferometer system.

2. A touch of reality can be given to this "physical" interpretation by considering one of the ways in which $\Lambda$ and $\Lambda$, can be computed in practice. Let the interval in which $\omega$ is non-zero be $0<\omega<\pi$. Then $\Lambda$ can be written

$$\Lambda = \int \frac{\Lambda(\tau)}{\sqrt{\lambda \pi}} e^{j\omega \tau} d\tau$$

which can be interpreted as the output at time $t$ of a linear filter whose input is $\omega$ and whose impulse response, $h(t)$, is given by

$$h(t) = \omega_{e} (\tau-t)$$

This is the matched filter for this waveform. Clearly the output of this filter as a function of time is precisely equal to $\Lambda$ for various values of $\tau$. If $\omega_{e}(\tau)$ is a narrow-band waveform, then $\Lambda(\tau)$ will also have the appearance of a narrow-band waveform, $\Lambda(\tau)$ is precisely the ordinary physical envelope of this waveform, i.e., can be obtained by following the matched filter with an envelope detector.
In the literature $P(\theta \mid w)$ has in general
been called the a posteriori probability in the
"random-phase case." This is an unfortunate
name since it has led to considerable confusion
with the already rather confused question of
"coherent" vs. "incoherent" radar. We have gone
through the argument leading to $P(\theta \mid w)$ in some
detail in the hope of pointing out that there is
really no connexion between these two ideas.

The receiver computing $\Lambda$, is every bit as co-
herent as that computing $\Lambda$ in the sense that complete
knowledge of the internal phase structure of the
expected received signal is assumed in each
case—it is only the initial phase or detailed
local range which is assumed random and equivo-
cal in $\Lambda$. And, of course, in neither case is there
any necessity that the transmitted signal have
some regular predictable phase structure ("co-
herence-pulse-to-pulse")—it is merely necessary
in each case that the receiver be aware a posterior-
i of what was indeed transmitted, and this is
a condition which, at least in principle, can always be satisfied in radar.

3.0 Simple Detection Situations[2]

In the proceeding sections we have discussed
the radar model which we have selected and the
role which the a posteriori probability plays as a
complete measure of our after-the-fact knowledge.
But, although the a posteriori probability, sup-
plemented perhaps with some decision method, rep-
resents a more or less complete solution to the
analysis problem, we must go further before we can
do radar synthesis. In particular we must con-
sider the quality of our a posterior knowledge and
the way it depends on the various system parameters.
There are many sorts of quality judgments which
might be applied. We propose to consider just
four:

1. The reliability of the detection or
determination that a target echo is
"there,"

2. The accuracy with which the parameters
of the target echo can be measured,

3. The possibility of ambiguities in the
determination of target echo parameters,

4. The extent to which two target echoes
present simultaneously or overlapping
can be resolved and measured separately.

Of these four, the first—reliability of detec-
tion—clearly underlies or precedes the other three.
In order to acquire some feeling for the detection
problem we shall first consider several situations
in which the a priori knowledge is, by assumption,
such that the other three quality judgments do
not apply.

3.1 The Canonic Detection Problem

As we have mentioned before, the question of
detection or decision brings up the problem of
value judgments, i.e., the relative "costs" to be
assigned to the various ways and degrees of being
wrong. Fortunately there are several simple sit-
uations from which it is possible to draw remark-
ably general conclusions of great power and util-
ity without having to get deeply involved in such
elementary subjects as value judgments and decision
criteria. The simplest of these is philosophi-
cally almost trivial and might be called the cano-
nic detection problem. We assume that it is
known a priori that only one of two possible tar-
get situations can ever occur—either there is no
target present at all so that the received signal
consists of noise alone, or else, one particular
known target is present so that the received sig-
nal consists of a known echo signal, (i.e., known
waveform, $\theta$, $\tau$, and $\omega$ plus noise). In this
case there are only two a posteriori probabilities
of interest—$P(\theta \mid w)$ and $P(\theta \mid w \neq \theta)$. We
have gone through the argument leading to $P(\theta \mid w)$ in some
detail in the hope of pointing out that there is
really no connexion between these two ideas.

Since our complete a posteriori knowledge of the
situation is thus specified by a single number,
$P(\theta \mid w)$, it is clear that the only rati
on decision process is a comparison of $P(\theta \mid w)$ with a threshold—announcing desired signal present if
$P(\theta \mid w)$ exceeds this threshold, and otherwise
absent. Moreover, since $P(\theta \mid x)$ is monotonic in
$\tau$, a completely equivalent process, and one
which is perhaps more easily interpreted, is merely
a comparison of $\Lambda$ with a different threshold, $\tilde{\Lambda}$.
The choice of $\tilde{\Lambda}$, of course, depends on the speci-
ified values of $\omega$, $\tau$, $\omega$, and on the ap-
propriate value judgment selected to rate the
performance of the decision process. But this is
the only way in which the questions of either the
value judgments or the a priori probability enter
the problem. Thus, in the a posteriori approach, in this case at least, is that we
state quite unequivocally that the form of the
optimum decision process, i.e., compare $\Lambda$
with a threshold, will not depend on the particu-
lar value judgment chosen, which is really quite
remarkable and important conclusion. Indeed if at
least some relative degree of invariance to such
an emotional quantity as value judgments were not
obtained in an a simple a decision problem as this,
we would have very serious doubts about the likeli-
hood of any really general and useful conclusions
coming out of the present approach.

The remaining result of interest in the
canonic detection problem is a determination of
those attributes of the received echo waveform
which influence the decision performance. First,
it is necessary to point out that the performance
of the detector in this simple problem can be com-
pletely characterized by two conditional probabil-
ities:

$P_a$ = probability of announcing echo signal
present if there actually is such a signal
present = Probability of Detection;

$P_f$ = probability of announcing echo signal
present if there actually is not a signal
present = Probability of False Alarm.

1. Clearly such a situation is almost too trivial
to ever be representative of an actual radar pro-
lem. Nevertheless, certain communications pro-
lems, e.g., synchronous PCM, are represented
rather accurately by this model.

2. Throughout this paper we shall assume that we
are dealing with the "random-phase case" so that
$P(\theta \mid w)$ rather than $P(\theta \mid w)$ is the appropriate
probability distribution.

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It is easy to show by an analysis of the statistical properties of $\Lambda$, that $P_d$ and $P_f$ are functions of just two parameters:

$$R = \frac{\Delta^2}{N_0} = \frac{\text{echo signal energy}}{\text{effective input noise power/ cps}}.$$  

$$\frac{\gamma^2}{N_0} = \text{effective input noise power/ cps}.$$  

The parameter $R/N_0$ can be eliminated and $P_d$ plotted as a function of $P_f$ with $R$ as a parameter. The resulting family of curves (Fig. 1) has been called the receiver detection characteristic.  

In accordance with the argument of the preceding paragraph the interpretation to be put on these curves is the following. For a given $R$ any pair of values of $P_d$ and $P_f$ lying on the corresponding curve can be obtained by comparing $\Lambda$ with an appropriate threshold. The best operating point is, of course, a function of the selected value judgment and, in general, $P_d(M, \tau, \omega)$. But in any case the performance so obtained is optimum in the sense that no pair of values of $P_d$ and $P_f$ above or to the left of this curve can be obtained by any means with the given $R$. Any actual receiver which fails to compute $\Lambda$ or its equivalent will yield operating points lying below this curve.  

We have found the curves of Fig. 1 very useful for computing the performance of radars and other systems (as compared, for example, with the rather nebulous empirical rule that "reliable" detection requires some signal-to-noise ratio to be greater than 1).  

But for our present purposes the most important conclusion of the last paragraph is that the detector performance, in the non-coherent (or scintillating) case, depends only on the ratio of the desired echo signal energy to the noise power per cycle per second, and not upon any other attribute of the waveform (e.g., bandwidth, waveshape, etc.). That the performance should in principle depend on the ratio, $R$ and not alone on the more common signal-to-noise power ratio is certainly not surprising in view of the equipartition Law of physics. But when translated into other terms, e.g., the observation that a pulse radar and a CW radar will have the same detection performance on a given target for the same average received power and observation time, despite the large difference in bandwidth, the argument is not always so readily believed.  

Before going on to consider more complicated decision problems it is worthwhile to investigate the effect of an unknown amplitude, $\Delta$, on what is otherwise the non-coherent decision problem. There are, perhaps, two extreme situations—one in which it is desired to estimate $\Delta$ as well as detect the presence of the signal, and the other in which the actual value of $\Delta$ contributes essentially no information and only a yes-or-no answer about presence is desired. The first situation brings up the question of accuracy; indeed, our discussion here will serve as the prototype for later discussions with respect to $v$ and $\omega$. The second situation provides another example of the proper way to handle a "stray" or non-information-carrying parameter. In the second case, in particular, it is necessary to make an assumption about the a priori distribution of amplitudes. For purposes of illustration we shall choose the Rayleigh distribution which in many cases is a reasonable approximation to the actual distribution and has the advantage of being easy to manipulate. That is we shall assume that

$$P_0(\nu, v, \omega) = \frac{v}{\nu^2} e^{-v^2/2\nu^2} v^2$$  

$$P(\nu, v, \omega) = \int_0^\infty P_0(\nu, v, \omega) 2\nu$$  

If only a yes-no answer about the presence of a target is desired, then paralleling the random phase arguments what we must compute is

$$\frac{P_0(\nu, v, \omega)}{P(\nu, v, \omega)} = \int_0^\infty P_0(\nu, v, \omega) 2\nu$$  

1. Indeed we consider the most meaningful and fundamental form of the radar equation to be that which relates $\nu^2$ rather than received signal power, to transmitted power, antenna gain, range, etc.

2. It should be remembered that we are still assuming that, whatever the value of $\Delta$, it remains constant for the duration of the signal. In physical terms we are thus modeling the slowly-scintillating target only. Rapidly-scintillating targets present much more difficult problems.
i.e., the sum of the a posteriori probabilities for each value of \( \rho \). The integration can be easily carried out and it can be shown that a comparison of \( P(\rho) \) with a threshold is completely equivalent to comparing \( \Lambda \) with a threshold as before. The difference from the case of known amplitude comes in the values of \( P_d \) and \( P_e \). The receiver detection characteristic is readily computed and the resulting family of curves is shown dotted in Fig. 1. The parameter in this case is the ratio \( \frac{\rho}{\rho_{\text{max}}} \), where \( \rho \) is the most probable target echo amplitude. The most significant attributes of these curves are the much lower values of \( \rho \) resulting for high values of \( R \) compared with the non-fluctuating case for the same value of \( \rho \), and the effective saturation in \( P_e \) accompanying an attempt to increase \( P_d \) by increasing \( \rho \). These results have an important influence on radar design, but further discussion of these effects is outside the scope of this paper.

Suppose, however, it is necessary to determine the actual amplitude of the target. The question then arises as to the accuracy with which \( \rho \) can be measured. The a posterior probability of a particular value, \( \rho \), is from (2-3)

\[
P(\rho) = \frac{e^{-\frac{\rho^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}
\]

(3.1-3)

Now as we have already shown the ratio \( \frac{\rho}{\rho_{\text{max}}} \) must be large if the signal is to be detected reliably. Or alternatively we shall show that \( \rho \) must be large if \( \rho \) is to be determined accurately. From either point of view we conclude that the interesting range of \( \rho \) in (3.1-3) is the neighborhood of \( \rho \), and that \( \rho_{\text{max}} \) and \( \rho_{\text{min}} \) will both be \( \gg 1 \). If \( P(\rho; \sigma^2) \) is continuous, we are probably entitled to assume that the variation of \( \rho \) in the neighborhood of \( \rho \) will be small so that the a priori probability may be effectively included for present purposes in the constant \( k \). If \( P(\rho; \sigma^2) \) presumably has a large peak at \( \rho_{\text{max}} \); the precise location of the maximum, i.e., the most probable value of \( \rho \), is determined by

\[
\frac{\partial P(\rho; \sigma^2)}{\partial \rho} = 0
\]

(3.1-4)

or

\[
\rho = \frac{\rho_{\text{max}}}{\rho_{\text{min}}}
\]

(3.1-5)

Using the fact that \( \rho_{\text{max}} \gg 1 \) and preserving only the first terms of the asymptotic expansions

1. Situations in which the amplitude of the return is a useful piece of information appear to be rare in radar systems. The commonest example, perhaps, is in monopulse systems, and here it is not so much the amplitude of the echo signal for a single radar as in effectively the ratio of the amplitudes for two radars which matters. It seems possible that future radar systems may perhaps find amplitude information useful as an aid to identification.

\[
I(\rho) \sim e^{-\frac{\rho^2}{2\sigma^2}} \left( 1 + \frac{\rho^2}{2\sigma^2} \right) \frac{1}{\sqrt{2\pi\sigma^2}}
\]

(3.1-6)

\[
I(\rho) \sim e^{-\frac{\rho^2}{2\sigma^2}} \left( 1 - \frac{\rho^2}{2\sigma^2} \right) \frac{1}{\sqrt{2\pi\sigma^2}}
\]

(3.1-7)

the solution of (3.1-5) is simply

\[
\rho = \lambda
\]

(3.1-8)

with an error the order of \( \frac{\lambda}{\rho_{\text{max}}} \). Equation (3.1-8) is certainly reasonable. In particular we note that as \( \rho \to \rho_{\text{max}} \).

We now wish to focus attention on a series of cases in which the actual amplitude is \( \rho \), and to consider the distribution of values of \( \rho \) which would result from (3.1-6). A study of the form of \( \rho \) (see (2-4) and footnote 1, page 2) shows that \( \Lambda \), has the same distribution as the envelope of a sinusoid of amplitude \( \rho \), plus a narrow-band Gaussian noise with variance \( \rho_{\text{max}} \). This problem has been considered by Rice, who has shown that the distribution for \( \rho \) is approximately normal (for \( \rho_{\text{max}} \gg 1 \)) with mean value equal to \( \rho \), as we should expect, and variance equal to \( \rho_{\text{max}} \). Thus the normalized effective standard deviation is approximately

\[
\frac{\sigma}{\rho} = \frac{1}{\sqrt{2\pi\sigma^2}}
\]

(3.1-9)

which is the result we sought. Typically \( \rho \) is the order of 100 or more so that a relative accuracy of better than 10 per cent in the determination of \( \rho \) is reasonable.

Completely aside from the various approximations employed, the procedure of the previous paragraph can be questioned on philosophical grounds. In outline what we did was the following:

1. Determined that value of \( \rho \), say \( \rho' \), for which \( P(\rho'; \sigma^2) \) had its maximum in the vicinity of \( \rho \); the peak of the maximum; the precise location of the maximum, i.e., the most probable value of \( \rho \), is determined by

\[
\frac{\partial P(\rho; \sigma^2)}{\partial \rho} = 0
\]

(3.1-4)

or

\[
\rho = \frac{\rho_{\text{max}}}{\rho_{\text{min}}}
\]

(3.1-5)

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this particular \( r(t) \). Now, of course, \( P(r, \omega) \) can tell us something general about the accuracy of estimating \( n \) if it \( P(r, \omega) \), in the vicinity of \( n \), at least, has essentially the same shape for nearly all received signals, \( r(t) \), having the same \( n \). But this basically implies that we can measure \( n \) with high accuracy which is exactly the condition we wish to achieve and are most interested in. Hence, the argument is self-consistent--if \( F(r, \omega) \) has a large spike in the neighborhood of some particular \( n \), i.e., if \( F(r, \omega) \) is exactly the probability that some \( n \) other than \( n \) could have been present in \( r(t) \). In such a case \( F(r, \omega) \) will be determined almost entirely by the echo signal and will be almost independent of the particular noise present so that we may make general statements. If \( F(r, \omega) \) does not have a large isolated spike then, although \( F(r, \omega) \) still measures the distribution of possible values of \( n \), that might have produced the particular \( r(t) \) received, the accuracy is presumably low and no general statements can be made since the noise has had a major effect on \( F(r, \omega) \). This same argument applies to any other parameter as well as to \( n \) and the one we shall employ for \( T \) and \( \omega \).

For the particular process being considered \( F(r, \omega) \) is given by (3.1-3). As before we shall assume that \( F(r, \omega) \) is essentially constant over the interesting neighborhood so that

\[
P(r, \omega) \sim e^{-\frac{r^2}{2b^2}} \tag{3.1-10}
\]

Now the function \( \Lambda \) can be broken up into two terms by writing \( r(t) = r(t) + w(t) \):

\[
\Lambda(t) = \int \sum_{s} S_s(t-x) \sum_{s} S_s(x) e^{j2\pi ft} dt + \int \sum_{s} S_s(t-x) \sum_{s} S_s(x) e^{j2\pi ft} dt \tag{3.1-11}
\]

The first integral is equal to \( \Lambda = \) amplitude of echo signal present. The supposition that \( F(r, \omega) \) has a large spike in the neighborhood being considered implies that the second term is with high probability small and that \( \Lambda / \Lambda \) is large so that

\[
P(r, \omega) \sim e^{-\frac{r^2}{2b^2}} \tag{3.1-12}
\]

\[
\sim e^{-\frac{r^2}{2b^2}} \tag{3.1-13}
\]

where we have replaced \( \Lambda(t) \) by the first term of its asymptotic expansion (3.1-6) and completed the square. Thus \( F(r, \omega) \) is approximately normally distributed near \( \Lambda = 0 \) with mean \( \mu = \max (F(r, \omega) - \Lambda) \), and standard deviation \( \sigma = \sqrt{\frac{\sigma^2}{\sigma^2}} \) as before. In addition to being simpler than the first approach, this method has, in Woodward's words "the remarkable feature----that [accuracy] can be studied in the absence of noise" since the effect of our argument was to remove the noise term from \( \Lambda \) and so far as computing \( F(r, \omega) \) was concerned.

In the preceding paragraphs we have considered nearly all the questions which can arise in the canonic detection problem, with the exception of those which more properly belong to the study of values and which determine the actual value of the threshold to be employed. To summarize we have observed that:

1. For the canonic detection problem the optimum form of decision process is a comparison of the envelope of the output of a matched filter or cross-correlator with a threshold. The form of this decision process is independent of either the type of value judgment selected or the a priori probabilities;

2. The reliability of detection, in so far as it depends on the characteristics of the radar and the target, depends only on the ratio of received signal energy to noise power per cycle per second, i.e., on \( R \);

3. The accuracy with which the parameter \( \omega \) can be determined is measured by the standard deviation of \( \Lambda / \Lambda \). For \( R > 1 \) (and we observed that \( \omega \) has to be much greater than one if the detection performance is to be reliable) the shape of \( F(r, \omega) \) in the neighborhood of the correct value of \( \omega \) is nearly the same for all \( \omega \) and is essentially equal to \( F(r, \omega) \) with \( \omega \) substituted for \( \omega \). Specifically the standard deviation of \( \omega \) is given by

\[
\sigma_{\omega} = \frac{\sigma}{\sqrt{R}}
\]

2.2 The Case of \( \Lambda \) Orthogonal Signals

We now wish to consider another simple detection problem which is somewhat more directly related to the radar problem than that discussed in the preceding section. As before we shall assume that at most one target echo signal is present at any one time or during any one observation interval. We shall further assume that the amplitude, \( A \), is known a priori and that the initial phase angle is random and informationless. However, unlike the preceding case we shall assume that the parameters \( T \) and \( \omega \) of the echo signal are not known a priori but instead any one of \( M \) signals of the form \( A e^{j\omega t} \) may be present with equal probability. We shall further assume that these \( M \) signals are mutually orthogonal, i.e., that

\[
\sum_{s} S_s(t-x) \sum_{s} S_s(t-x) dt = \begin{cases} 2 \pi \beta^{2} & \text{if } s = s' \\ 0 & \text{if } s \neq s' \end{cases}
\]

where the star indicates complex conjugate. Although this set of assumptions is clearly a closer approximation to actual radar problems than the canonic detection problem in that \( T \) and \( \omega \) are treated as unknowns, the assumed discrete nature of the a priori distributions in \( T \) and \( \omega \), and
the requirements that (3.2-1) be satisfied, are obviously unrealistic. The next section will be largely devoted to removing these restrictions, but their value for the moment is that again we will be able to say something of general value about the reliability of detection without getting involved in the accuracy, ambiguity, and resolution questions. Indeed we hope eventually to be able to conclude—and this is one of the principle points of our detection philosophy—i.e., that the radar detection problem breaks down into two essentially independent problems. The first of these is to adjust the radar parameters, particularly those which appear in the radar equation, in order to obtain sufficient received energy under the desired conditions to produce reliable detection of the fact that a signal is present. The important point is that the required signal energy, or better the required value of $R$, can be stated almost independently of the received waveform.

The second problem, then is to adjust the received waveform, by means, of course, of choosing the transmitted waveform, in order to obtain the required performance in terms of accuracy, ambiguity, and resolution.

For the moment, then, we are interested in the reliability of detection if there are $M$ orthogonal signals which might possibly be present one at a time, instead of just one known signal. From the beginning it is apparent that the present problem is philosophically considerably more involved than the simple canonic problem in that the a posteriori probability distribution (which is still given by (2-7)) now consists of a set of $M$ numbers instead of just two. In particular it is no longer possible to circumvent more or less completely the question of value judgments—there are many meaningful and different ways in which the question "Is there a signal present?" can be asked. For our present purposes, however, as outlined in the preceding paragraph, it will perhaps suffice to demonstrate the essential invariance of the system behavior by analysing two examples.

The first of these is perhaps the most obvious formulation of the pure detection problem in the present situation. If we reduce the detection problem to a simple binary decision then the performance can be completely described before by the various probabilities of being right and wrong, such as $P_d$ and $P_f$. In particular for our first example we seek the optimum receiver detection characteristic relating

$$P_d = \text{probability of announcing that a (any) signal is present when indeed there is a (some) signal present.}$$

and

$$P_f = \text{probability of announcing that a (any) signal is present when actually only noise is present.}$$

From the discussion in preceding sections it should be obvious that the corresponding optimum decision operation is to compare

$$y_i = f(\alpha, \omega, \omega')$$

with a threshold. Using the assumption of orthogonality (which assures statistical independence among the terms of the sum) and assuming that $M$ is large enough so that the central limit theorem of statistics may be employed, it is a more or less straight-forward problem to show that the receiver detection characteristic has essentially the form of the solid lines of Fig. 1 where the parameter, $R$, is to be interpreted, not as equal to $\pi^2/6$ but rather

$$R = \frac{\pi^2}{6} - \ln M$$

(3.2-2)

Actually the approximations are such that this expression for $R$ is slightly pessimistic.

The most important conclusion to be derived from this result is that the value of $\pi^2/6$ required for a given performance is only logarithmically dependent on $M$. Indeed we would be justified in stating that to a first approximation the required value of $\pi^2/6$ is independent of $M$. The essential truth of this statement is perhaps best illustrated by an example. For "reliable" detection (e.g., $P_d = 0.99$, $P_f = 10^{-5}$) the $R$ of Fig. 1 typically must be the order of 50. If $M = 20,000$ then $\pi^2/6$ must equal 60 which is an increase of just 0.8 db over the Value of $\pi^2/6$ ($\approx 50$) which would be required for the same $P_d$ and $P_f$ if $M = 1$, i.e., if $\omega$ and $\omega'$ were fixed and known a priori.

As a second example we consider a decision process of a somewhat different nature. We postulate the existence of a receiver with $M$ output channels, one for each possible echo signal. Each channel has two output states corresponding to "signal present" and "signal absent" and we suppose that the channels are so interlocked that only one channel can indicate "signal present" at a time.

We then seek the optimum receiver detection characteristic relating

$$P_d = \text{probability that a particular signal will be announced if indeed that particular signal is present,}$$

and

$$P_f = \sum_{i=1}^{M} P_{i}$$

where $P_{i} = \text{probability that the } i \text{th signal will be announced if indeed the } i \text{th signal is not present.}$

Although this criterion is obviously quite a bit stiffer than that discussed in the first example, we cannot argue that the resulting detector will be better or worse than the first detector unless the use to which the data is to be put and the corresponding appropriate value judgments are considered. Our purpose, however, is not to compare these criteria but rather to show that even in this second case a large increase in $M$ requires only a small increase in the value of $\pi^2/6$ to keep the reliability of detection constant. It is easy to argue that the optimum detection process

1. The probability of each of the $M$ signals together with the probability, $P(0,0,0)$ of noise alone.
in this second case is to compare \( A \), separately
with a threshold for each signal; assume
that signal present, if any, for which the corresponding
\( A \) is in excess of the threshold. As a re-
result of the assumed orthogonality among the sig-
als, the solid curves of Fig. 1 may then be inter-
preted as plotting the relationship between \( P \) and \( P' \).
Assuming that all signals are treated alike

\[
P_f = \frac{M}{P_f}
\]

To illustrate (using the same example before)—
suppose \( P_f = 0.99 \), \( P_f = 10^{-5} \), \( M = 20,000 \).
Then \( P_f = 5 \times 10^{-10} \) and the required value of \( \frac{P}{P_f} \)
is \((8.8)^2 = 77.4 \) which is a 1.9dB increase over
the value of \( 8 \). Required for \( P \) and \( \omega \) known
\( \text{a priori} \) and a 1.1dB increase over the value re-
quired in the first example—in neither case a
very significant amount considering the size of \( M \).

4.0 Detection in the Case of Continuous Parameter Distributions: Accuracy
and Ambiguity.

As soon as we pass from the discrete \( \text{a priori} \) dis-
tribution assumed for \( \tau \) and \( \omega \) in the pre-
ceding sections to more realistic continuous dis-
butions, a whole host of problems arises,
these problems are not only of a mathematical or
computational nature but also, as we saw in a
rather elementary way in the example of section
3.2, involve quite complicated questions of value
judgments and problem formulation. Roughly the
difficult is that it is no longer possible to
state performance measures in black and white
terms; there are various ways or degrees of being
wrong and the penalties must be weighted accoun-
tingly. Nevertheless, if we have any hopes of
Evolving a useful theory we must face these pro-
blems, even if our conclusions are more qualita-
tive than quantitative.

As before we shall assume that at most one
target echo is present during any observation
interval and that the initial phase angle of the
return is random and informationless. We shall
further assume, for simplicity and to be definite,
that:

1. The amplitude, \( A \), is known and constant,
independent of \( \tau \) and \( \omega \).

2. The \( \text{a priori} \) probability density \( P(\tau,\omega) \)
is a constant for all values of \( \tau \) and \( \omega \)
lying inside the rectangle in the \( \tau,\omega \) plane
bounded by the lines \( \tau = \tau_0 \), \( \omega = \omega_0 \),
\( \tau = \tau_1 \), \( \omega = \omega_1 \).
In other words all pairs of values of \( \tau \)
and \( \omega \) satisfying the conditions \( \tau_0 \leq \tau \leq \tau_1 \)
and \( \omega_0 \leq \omega \leq \omega_1 \) will be assumed equally likely.

Other, less restrictive, assumptions can be hand-
led, at least qualitatively, but those will serve
to give the principal outlines of what can be done.
Loosely, we shall be concerned with three
questions. The first of these is essentially the
same question considered previously, i.e., what
is the reliability of detection, where by detec-
tion we have in mind essentially the same sort of
decision as in the first example of section
3.2. And, indeed, our method of handling this
problem will be to replace the actual continuous
parameter situation by an approximately equivalent
discrete orthogonal problem of the type analyzed
in section 3.2. The other two questions are new.
One is the question of the accuracy with which
the parameters \( \tau \) and \( \omega \) can be measured once it
has been ascertained that an echo is indeed pre-
sent. We have considered the question of accuracy
before with respect to the measurement of ampli-
tude in the canonical detection problem. Essential-
ly the same method will be employed here for \( \tau \)
and \( \omega \). The third question concerns the pos-
sibility of ambiguity, i.e., are there other val-
ues of \( \tau \) and \( \omega \) significantly removed from the
proper values which might conceivably be mis-
construed as the right values. Actually there
are two ways in which an ambiguity might arise.
One possibility is that the noise accompanying
a particular echo might look sufficiently like
some other echo that the \( \text{a posteriori} \) probability
of this latter signal might be large. This type
of ambiguity really has more the character of a
false alarm—if the detection is "reliable" than
such ambiguities should be rare. On the other
hand the structure of the signal may be such that
two echoes from different targets may look much
alike, e.g., the "second-time-around" target in
conventional pulse radar. This is the type of
ambiguity we wish to study. The principal ob-
jective of our study, of course, is radar synthe-
sis. Hence, with respect to accuracy and ambig-
ity we shall seek both for those attributes of
the radar which influence these problems and for
appropriate limit theorems or realizability con-
ditions on the types of accuracy and ambiguity
performance which can be obtained. We shall find,
of course, that unlike the reliability of detec-
tion problem, the important parameters influen-
cing accuracy and ambiguity are those which de-
scribe the waveform, and we shall discuss an im-
portant performance constraint on the waveform
which perhaps deserves the title of the Radar
Uncertainty Principle.

As a result of the various assumptions which
we have made, the \( \text{a posteriori} \) probability density
for the present situation can be written in a some-
what simpler form than (2-3). Specifically,

\[
\rho(\tau,\omega) = \kappa \int \frac{A^2}{(A^2 + \lambda)^2} d\lambda
\]

where we have absorbed into \( \kappa \) a multitude of
terms including the \( \text{a priori} \) probability density.
Now it should be obvious that if we are going to
tackle the reliability of detection in the present
discrete orthogonal problem of the type analyzed
in section 3.2, i.e., in terms of

\[
P_d = \text{probability of announcing that a (any)}
\text{signal is present when indeed there is a (some)}
\text{signal present;}
\]

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and $P_F$ = probability of announcing that a (any) signal is present when actually only noise is present; then by analogy with that example the optimum detection procedure is to compare
\[ \Phi_{d} \] with a threshold. We can even perhaps imagine an infinite number of cross-correlators followed by non-linear weighting devices, a summing circuit, and a comparison circuit which would physically carry out this operation. But a real difficulty arises when we try to compute $P_d$ and $P_F$ since the noise outputs of these individual channels will not in general be independent—the corresponding signals will not in general be orthogonal or uncorrelated.

The question of the way in which the various possible echo signals are correlated is a most important one for our study since it not only influences the value of $\psi$ required for a given reliability of detection, but also has a major effect on the accuracy and ambiguity question. To see this let us consider two signals $\Psi_1(t, \tau, \omega)$ and $\Psi_2(t, \tau, \omega)$, and let us then compute both $\lambda_{d}(t, \tau, \omega)$ and $\lambda_{f}(t, \tau, \omega)$ in the case in which $\Psi_2(t, \tau, \omega)$ is actually present
\[ \lambda_{d}(t, \tau, \omega) = \frac{1}{P} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} n(t, \tau, \omega) S_2(t, \tau, \omega) e^{j\omega t} dt \] \[ \lambda_{f}(t, \tau, \omega) = \frac{1}{P} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} n(t, \tau, \omega) S_2(t, \tau, \omega) e^{j\omega t} dt \] If, as we have argued before, the detection is to be reliable, then the first term in (4-3) must be much larger than the second so that
\[ \lambda_{d}(t, \tau, \omega) \approx \Phi \] (4-5)
If in addition $\Psi_1(t, \tau, \omega)$ and $\Psi_2(t, \tau, \omega)$ are highly correlated, by which we mean that
\[ \frac{1}{Z} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (t, \tau, \omega) S_1(t, \tau, \omega) dt / \omega \] then it will also be true that
\[ \lambda_{d}(t, \tau, \omega) \approx \Phi \] (4-6)
Thus
\[ \lambda_{d}(t, \tau, \omega) \approx \Phi \] (4-7)
and so
\[ \lambda_{f}(t, \tau, \omega) \approx \Phi_{d}(t, \tau, \omega) \] (4-8)
whether $\Phi_{d}(t, \tau, \omega)$ will actually be greater than or less than $\Phi_{d}(t, \tau, \omega)$ will depend on the particular noise waveform present, but will not depend very much on which signal is present. In other words, when $\Psi_1(t, \tau, \omega)$ is actually present, we can not really be sure if it is $\Psi_1(t, \tau, \omega)$ or $\Psi_2(t, \tau, \omega)$ which is present. Thus if two possible received waveforms are highly correlated in the sense of (4-8) and one of them is present, the determination of which one is present is fundamentally ambiguous and no amount of clever data processing can resolve this ambiguity.

In a similar situation suppose that for some fixed value of $\omega = \omega_0$ and for all values of $\tau$ lying in some interval $\Delta \tau$ centered on $\tau$, the corresponding signals $\Psi_1(t, \tau, \omega)$ are highly correlated in the sense of (4-9). Then if one of these signals is present we shall not be able to determine which one, i.e., we shall not be able to measure $\tau$, with an accuracy greater than the order of $\Delta \tau$. This, of course, is in essence the same argument as we employed in section 3.1 with relation to the accuracy of estimating $\tau$.

To be sure the accuracy and ambiguity situations depend not only on the degree of correlation of the various signals but also on the ratio $\psi/n$ --we shall have more to say about this in what follows. But the important point is that the limiting possible performance with respect to accuracy and ambiguity depends not on ingenuity in processing the received signals but rather on the shape of the received signals themselves and in particular on the extent to which the various received signals are correlated. It behooves us, therefore, to study the various possible forms which this correlation can take. Such a study will permit us not only to give more-or-less complete answers to the detectability, accuracy, and ambiguity questions in particular cases but will lead to one of the most important theorems constraining radar performance--the Radar Uncertainty Principle.

4.1 Waveform Examples; Radar Uncertainty Principle

We are interested in the behavior of the function
\[ \lambda_{d}(t, \tau, \omega) = \int_{-\infty}^{\infty} (t, \tau, \omega) S_1(t, \tau, \omega) dt / \omega \] (4.1-1)
\[ \lambda_{f}(t, \tau, \omega) = \int_{-\infty}^{\infty} (t, \tau, \omega) S_2(t, \tau, \omega) dt / \omega \]
and the second expression has been obtained from the first by an elementary change of variable. Physically \( \mathcal{S}(t_1, t_2) \) can be loosely interpreted as the output in the absence of noise of a cross-correlator corresponding to a particular signal when a second signal with a delay less by \( t' \) and a frequency shift less by \( \omega' \) than that particular signal is present. Alternately \( \mathcal{S}(t_1, t_2) \), considered as a function of \( t_2' \), and with \( t'_2 \) interpreted as time, is, except for a scale factor and ignoring noise, precisely the time waveform corresponding to the envelope of the output of a matched filter for a signal at a particular frequency when a signal at a frequency \( \omega' \) less is present. Although \( \mathcal{S}(t_1, t_2) \) has a number of interesting and important mathematical properties, it is perhaps more expedient for our present purposes to proceed to a consideration of several examples.

### 4.11 Impulse or CW Radar

We shall assume for our first example that the received waveform is a single pulsed-sinusoid of constant amplitude and duration, \( T \). Thus, recalling the normalization of equation (1.2)

\[
\mathcal{S}(t) = \begin{cases} \frac{1}{\sqrt{T}}; & 0 \leq t \leq T \\ 0; & \text{elsewhere} \end{cases}
\]  

(4.11-1)

Physically there are two interesting extreme situations to which this example might correspond.

- **CW Radar**—where \( T \) is essentially the "time on target" and typically \( T \gg T' \), but \( \frac{2\pi}{\omega} \ll \chi \\

- **Impulse Radar**—where \( T \) is now simply the impulse duration and typically \( T' \ll T \), but \( \frac{2\pi}{\omega} \gg \chi \\

\( \mathcal{S}(t_1, t_2) \) is shown in Fig. 2 plotted as amplitude above the \( t_1-t_2 \) plane. We note that \( \mathcal{S}(t_1) \) is the highest point in the plane—which in reasonable since \( \mathcal{S}(t_1, t_2) \) is proportional to the output of the cross-correlator when the corresponding signal is present. Indeed we could prove the general result that for any waveform \( \mathcal{S}(t) \) of finite duration

\[
\mathcal{S}(t_2) \gg \mathcal{S}(t_1, t_2) ; \quad t_2 \ll t_1
\]  

(4.11-1a)

Philosophically this equation can be interpreted to mean that if the noise is vanishingly small, the parameters of an echo signal can be determined with perfect accuracy and without ambiguity—a physically satisfying conclusion.

An alternative way of representing \( \mathcal{S}(t_1, t_2) \) is shown in Fig. 3 and 4 where we have chosen the \( t_1 \) and \( t_2 \) scales so as to more clearly illustrate the difference between \( \omega \) and \( \omega' \) in this paper we have chosen to indicate the magnitude of \( \mathcal{S}(t_1, t_2) \) as the density of shading in the two-dimensional \( t_1-t_2 \) plane. Moreover, for simplicity we have somewhat arbitrarily restricted the degrees of shading to just three—black corresponding to highly correlated regions, i.e., \( \mathcal{S}(t_1, t_2) \approx 1 \), cross-hatch corresponding to weakly correlated regions, i.e., \( \mathcal{S}(t_1, t_2) \approx 0 \) unshaded corresponding to uncorrelated regions, i.e., \( \mathcal{S}(t_1, t_2) = 0 \).

Let us now consider the problem of determining the receiver detection characteristic for the case, say, of the CW radar of Fig. 3. We first note that it is not necessary to provide the channel in our detector for every possible pair of values of \( t_1 \) and \( t_2 \) as we assumed earlier. For example, since \( t' = T \) the channel corresponding to \( t_1 = 0 \) and any particular \( t_2 \) will have an output under all circumstances which is very highly correlated, i.e., nearly identical, with the output of the channels for the same \( t_2 \) and any \( t_1 \). Thus we could replace all these channels in the integration (4-2) with just one channel, say \( t_1 = 0 \). Similarly in frequency except channels separated \( \omega_1 \) from a given channel are nearly uncorrelated with the given channel. Thus approximately \( \omega_1 \) channels are required in frequency to cover the expected range of targets and the outputs of these channels are nearly orthogonal. Thus finally the detection performance in the case of a CW radar cannot be very different from that of the first example of section 3.2 with \( \omega = \omega' \).

Of course we could easily be off by a factor of 2 or more in this value of an equivalent \( M \) but since the required value of \( \omega_1 \) depends on \( M \) only logarithmically such an error has negligible significance. Moreover the whole argument is somewhat academic since, as we showed in section 3.2 the increase in \( \omega_1 \) required even for a large value of \( M \) is quite small. Of course a precisely dual argument holds in the case of the impulse radar, the required value of \( M \) being the order of \( \frac{2\pi}{\omega} \).

Next we shall examine the accuracy of \( \mathcal{S}(t_1, t_2) \) with which the parameters \( \omega \) and \( \omega' \) can be determined in the case of a CW radar. Of course, since for the allowed variation in \( t_1 \), all possible signals are highly correlated, essentially no estimate of \( \omega \) can be given unless \( \mathcal{S}(t_1, t_2) \) is very large, as is physically obvious. Measurements of \( \omega \) are more interesting. Using the same argument as in section 3.1 the procedure is to identify the variance in the measurement of \( \omega \) with the mean square width of the spike in \( \mathcal{S}(t_1, t_2) \) computed in the absence of noise. As should be readily apparent this is precisely the same as the mean square width of the spike at the origin of \( \mathcal{S}(t_1, t_2) \).

Making the usual approximations we obtain

\[
\omega \approx \frac{2\pi}{\sqrt{t_1-t_2}}
\]  

(4.11-2)
as the standard deviation of the error in \( \omega \) assuming \( \Delta X / X > 1 \). If we take \( \Delta X / X \) as the width in frequency of the central spike in \( \psi(\omega, \phi) \) and if we assume \( \Delta X / X = 50 \) then (4.11-2) states that we should be able to determine \( \omega \) to about 1/10 of this width. This is somewhat better than we are able to achieve in practice because of the effect of systematic errors which have been ignored. Equation (4.11-2) is a special case of a general result which states that

\[
\omega = \frac{1}{\sqrt{\bar{T}}} \rho
\]

(4.11-3)

where \( \rho \) is the root mean square duration about the mean of the signal. An essentially similar result holds in general for time measurements,

\[
\sigma_T = \frac{\sqrt{\bar{T}}}{\sqrt{\pi}} \beta
\]

(4.11-4)

where \( \beta \) (radians/sec) is the root mean square bandwidth about the mean frequency of the signal.

Unfortunately the approximations on which this latter formula is based breakdown for a square pulse. Using a slightly modified procedure one obtains the formula

\[
\sigma_T = \frac{\sqrt{\bar{T}}}{\sqrt{\pi}} N
\]

(4.11-5)

which is probably a trifle optimistic, even in theory.

### 4.12 Linear-Sweep FM-CW Radar

The possibilities with the preceding example were rather limited. Since the bandwidth for a single pulsed-sinusoid is roughly just the reciprocal of the duration it is generally not possible to find a value of \( T \) which will simultaneously give acceptable accuracy in both range and velocity. The obvious strategy, then, is to modulate the signal so that the bandwidth can be made many times larger than the reciprocal of the duration. A simple possibility would seem to be to frequency modulate the signal, e.g., to let

\[
S(t) = \begin{cases} \sqrt{2} e^{j \frac{4\pi t}{T}} & \text{if} \; t < T \\ 0 & \text{elsewhere} \end{cases}
\]

(4.12-1)

Thus the frequency of the pulse increases linearly from \( \omega_0 \) at \( t = 0 \) to \( \omega + \Delta \omega \) at \( t = T \). \( \psi(\omega, \phi) \) is readily computed for this waveform and the result can roughly be represented as in Fig. 5. As we can see our strategy has not been entirely successful. To be sure, we observe that we can make a measurement of \( T \) with an accuracy the order of \( \Delta \omega / \omega \) (roughly an improvement of \( 10^6 \) over a pulsed-sinusoid of the same duration) but only if we know the proper value of \( \omega \). And conversely we can determine \( \omega \) to an accuracy the order of \( \Delta T / T \) if we know the proper value of \( T \).

But if we know neither \( T \) nor \( \omega \) the best we can determine is a relationship between \( T \) and \( \omega \) of the form \( T^2 - \omega^2 = C \), constant dependent on signal received. Of course this result is hardly unexpected from a physical point of view since the waveform resulting from a frequency shift and from an appropriate time delay are very similar. Compared with a pulsed-sinusoid of the same duration, what we had hoped to achieve by frequency modulation was a compression in \( T \) and a rotation in \( \phi \) rather than a compression in \( \omega \) and velocity information is of little use. In this case FM-CW has essentially the accuracy and freedom from ambiguities of an impulse radar having the same bandwidth, together with the lower peak power for the same energy characteristic of the CW case. Moreover, assuming that we can guarantee that we are looking at one and the same target, it may be possible to make a second measurement with a different \( \omega \) (e.g., \( -\Delta \omega \)) and thus to determine both \( T \) and \( \omega \) with high accuracy and without ambiguity. Nevertheless, compared with other waveforms to be considered, the price of the FM-CW radar is high in terms of bandwidth for the results obtained. Its principle advantage is simplicity of implementation.

### 4.13 Coherent Periodic Pulse Radar

The commonest radar waveform, of course, is the periodic pulsed-sinusoid with or without various minor variations. We shall assume for analysis that \( S(t) \) is real and has the form indicated in Fig. 6a. By implication we are thus assuming that carrier phase is coherent from pulse to pulse, i.e., that the pulses are merely bursts selected from the same continuous sinusine. A closely related waveform is generated physically by starting an oscillator from noise separately for each pulse so that the carrier phase is random from pulse to pulse. The performance obtained with this second, incoherent, waveform is slightly different in some respects from that to be described as will be mentioned in a later section.

\[
\psi(\omega, \phi)
\]

for the waveform of Fig. 6a is shown in Fig. 6b and 6c. Clearly the accuracy of simultaneous measurements of \( T \) and \( \omega \) is now essentially the best which can be expected for a waveform of this bandwidth and total duration. More exactly, equations (4.11-2) and (4.11-5) may be applied to this case yielding

\[
\sigma_T = \frac{\sqrt{\bar{T}}}{\sqrt{\pi}} N
\]

(4.13-1)

\[
\sigma_\omega = \frac{\sqrt{\bar{T}}}{\sqrt{\pi}} \beta
\]

(4.13-2)

1. Clearly the best way to describe this situation would be in terms of the parameters of an ellipse in the \( T-\omega \) plane.
Fig. 2: $\psi(\tau, \omega)$ for Single Pulsed-Sinusoid.

Fig. 3: $\psi(\tau, \omega)$ for CW Sinusoid.

Fig. 4: $\psi(\tau, \omega)$ for Impulse of Sinusoid.

Fig. 5: $\psi(\tau, \omega)$ for Linear-Sweep PLL-CW.

Fig. 6: $\psi(\tau, \omega)$ for Periodic Pulsed Sinusoid.

Fig. 7: "Ideal" $\psi(\tau, \omega)$ for Radar.

Fig. 8: $\psi(\tau, \omega)$ for Noise Waveform.
where $\tau$, $\Delta$, and $\mathcal{W}$ have the significance indicated in Fig. 6a and $\mathcal{P}$ is to be interpreted as the total received energy.

Although the accuracy situation, thus, leaves little to be desired in that essentially the best possible performance is obtained within the allowed bandwidth, apparently a new difficulty has appeared. Assuming that $\mathcal{P} > \mathcal{A}$ and/or $\omega > \omega_0$, there will now be ambiguities in the determination of $\tau$ and/or $\omega$. These correspond to the familiar "second-time around" echoes and "blinds velocities." It is true, of course, that the spikes in $\mathcal{P}(\tau,\omega)$ at multiples of the repetition rate and repetition interval are smaller than the spike at the origin, and hence if the ratio $\rho = \frac{\mathcal{A}}{\mathcal{P}}$ is large enough, the highest spike in $\mathcal{P}(\tau,\omega)$ should with high probability correspond to the correct parameters. Some idea as to how large $\rho$ must be can be acquired from the following argument. Suppose that there is only one possible alternate pair of values which might be confused with each target. Such a situation might arise, for example, if the a priori information made it possible to discard other alternatives as unlikely or impossible, e.g., $\mathcal{Q} \rightarrow \mathcal{A}$ and/or $\omega < \omega_0$.

Let $\mathcal{P}(\tau',\omega')$ be the height of the ambiguous spike. We wish, then, to compute the probability that the wrong spike in $\mathcal{P}(\tau',\omega')$ will actually be higher (because of noise) than the correct spike. Clearly, if $\mathcal{P}(\tau',\omega')$, then there is complete equivocation; both spikes in $\mathcal{P}(\tau',\omega')$ will under all conditions be identical in height and we can say that the probability of error is 0.5. We can compute the relationship between $\rho$ and $\mathcal{P}(\tau',\omega')$ such that the probability of error is less than some arbitrary value, say 0.1. The result is approximately

$$\mathcal{P}(\tau',\omega') < 1 - \frac{\tau'}{\rho}$$

(4.13-3)

Thus if $\mathcal{P}(\tau',\omega') = 0.97$, $\rho = \frac{\mathcal{A}}{\mathcal{P}}$ would have to be greater than 115 before the probability of error would be less than 10%. Of course (4.13-3) is a theoretical result and ignores such questions as distortion and drift. Independent of the value of $\rho$, it would be necessary to go to great deal of trouble to build practical equipment capable of distinguishing reliably and over a wide dynamic range a difference as small as 3%. For most practical purposes spikes in $\mathcal{P}(\tau',\omega')$ greater than, say, 0.5, constitute unresolvable ambiguities.

Finally the presence of ambiguities has a significant effect on the value of $\mathcal{M}$ to use in equation (3.2-2) for estimating the reliability of detection. If both $\mathcal{Q} \rightarrow \mathcal{A}$ and $\omega < \omega_0$, then it is easy to argue that the appropriate value of $\mathcal{M}$ is roughly

$$\mathcal{M} \approx \frac{\rho}{\frac{\tau'}{2}} = \frac{\rho - T}{\frac{\tau'}{2}}$$

(4.13-4)

On the other hand if $\mathcal{Q} \rightarrow \mathcal{A}$ and $\omega > \omega_0$, then not all of the signals given by (4.13-4) will be independent.

1. Actually (4.13-3) applies to the case of known initial phase angle, but the error in $\mathcal{M}$ resulting from applying (4.13-3) to the random initial phase case is small.

i.e., equal to the time-bandwidth product for the signal. Considering the implications of the Sampling Theorem concerning the number of degrees of freedom in a signal of limited time and frequency duration, this is an entirely reasonable result.

4.14 The Radar Uncertainty Principle

It should at this point be obvious that $\mathcal{P}(\tau,\omega)$ corresponding to the ideal radar waveform should have the appearance of Fig. 7—a single narrow spike at the origin and nothing anywhere else in the plane. For maximum accuracy the spike should have a width of approximately $\frac{T}{\mathcal{W}}$ in frequency and $\frac{1}{\mathcal{W}}$ in time, and $\mathcal{P}$ and $\omega$ should be independently adjustable. The difficulty is— as we might expect—that such conditions are fundamentally impossible to achieve. We now wish to study why.

Our efforts thus far to achieve a waveform having a $\mathcal{P}(\tau,\omega)$ similar to Fig. 7 have been of like squeezing a pillow—as we push in one direction the pillow bulges out in the other, and if we are too persistent the casing breaks and we have piles of feathers all over the landscape. Or perhaps a better analogy would be to imagine that the $\mathcal{P}(\tau,\omega)$ contour is the surface of a pile of sand. As we adjust the waveform we seem to be able to move the sand around but unable to get rid of any of it. This latter analogy, with one modification—namely that we should talk about the $\mathcal{P}(\tau,\omega)$ contour instead of the $\mathcal{P}(\tau,\omega)$ contour, is actually a precise statement of the most important limitation to radar performance, i.e., what we shall call the Radar Uncertainty Principle. But before we give a precise formulation of this principle it is perhaps valuable to demonstrate it in an approximately quantitative manner for the various waveforms we have thus far considered. For most radar waveforms $\mathcal{P}(\tau,\omega)$ consists roughly of a number of spikes of approximately unit height together with regions in which $\mathcal{P}(\tau,\omega) = 0$. An approximate evaluation of the volume under the $\mathcal{P}(\tau,\omega)$ contour can be achieved by replacing the spikes with roughly equivalent cylinders of unit height and ignoring the volume in the regions where $\mathcal{P}(\tau,\omega) = 0$. For the three waveforms thus far considered this crude volume computation is as follows:

- **[base Area of Unit] x [Approximate No. of Cylinders] = Volume**
  - CW: $[(\tau) \times (2\pi)] \times [\frac{1}{2\pi}] = 2\tau$
  - **Fm**: $[(\tau) \times (2\pi)] \times [\frac{1}{2\pi}] = 2\tau$
  - **Radar**: $[\Delta] \times (2\pi) \times [\frac{1}{2\pi}] = 2\Delta$

Although it is something of an accident that this rather crude method works so nicely in these cases,
the conclusions nevertheless are correct. More formally, a precise statement of the Radar Uncertainty Principle is:

\[
\int_{-\infty}^{\infty} |\psi(t,\omega)|^2 dt = \int_{-\infty}^{\infty} |\psi(t,\omega)|^2 d\omega = 1
\]  

(4.14-1)

This result is easily proved by directly carrying out the indicated integrations after substituting the definition of \(\psi(t,\omega)\) from (4.1-1).

The Radar Uncertainty Principle has a number of important applications:

1. As a part of the postmortem probability approach, the Uncertainty Principle helps to emphasize that waveform selection, rather than ingenuity in detector design, is the determining element in radar accuracy, ambiguity, and (as we shall see) resolution performance.

2. By setting a bound on performance quality, it prevents much fruitless searching for waveform and detection methods intended to achieve such impossible performance as that described at the beginning of this section.

3. The Radar Uncertainty Principle has proved very helpful in finding the flaws in various suggested radar waveforms, detection procedures, MTI schemes, etc. Specifically, one can be sure that the analysis is complete if and only if all of the volume under the \(\psi(t,\omega)\) contour has been accounted for.

Unfortunately, although the Radar Uncertainty Principle represents a necessary condition for the existence of a waveform \(\psi(t,\omega)\) having a given form, it is not sufficient. A number of additional conditions can be specified, including sufficient conditions, but the forms of these conditions are not sufficiently simple to be really useful in waveform synthesis. There seems to be no real substitute at this point for an educated but intuitive guess followed by careful analysis.

4.15 The Ideal Waveform

We shall conclude this section with a discussion of several waveforms which come as close as possible to the ideal radar waveform—at least from the standpoint of accuracy and freedom from ambiguities. These waveforms have essentially the \(\psi(t,\omega)\) plot of Fig. 8, i.e., a single central spike of width \(\Delta f\) in frequency and \(\Delta f\) in time (where \(\omega\) and \(\tau\) are the echo duration and bandwidth in rad/sec respectively) with the remainder of the necessary volume (if \(\tau\) is large this will be nearly all the volume) spread out more-or-less uniformly over a region roughly \(\tau\) wide in time and \(\omega\) wide in frequency.

All of the waveforms corresponding to Fig. 8 have in common that they are in some sense noisy or pseudo-random—by which we mean that they take many numbers to specify them opposed to the waveforms we have already considered which are specified by just a few numbers, e.g., pulse length, repetition rate, etc. For example, if \(\tau\) is a sample from almost any sort of noise, e.g., hard-limited narrow-band Gaussian noise of duration \(\tau\) and bandwidth \(\omega\), the corresponding \(\psi(t,\omega)\) will with high probability look like Fig. 8 provided that \(\tau\) is very large, the order of \(10^7\) to \(10^9\) or so. But when \(\tau\) is only the order of \(10^4\) to \(10^5\), the noise waveform has to be selected with some care if spikes of height 0.5 or more at values of \(\tau\) and \(\omega\) other than the origin are to be avoided.

An interesting example of a suitable waveform having \(\tau\) = 0 or less is the coded-pulse waveform. This waveform is constructed by starting with a pulsed sinusoid of duration \(\tau\). This pulse is then divided into \(\tau\) intervals, each of duration \(\Delta\tau\). Each interval is then preserved as originally, or reversed in phase by 180°, according to whether the corresponding position in a binary sequence or code of length \(2^m\) is 0 or 1. Clearly the performance of such a waveform then depends on the code selected. Almost any pulse-modulated waveform having a bandwidth \(\omega\) or less can be closely approximated by choosing the proper code. At the moment we are interested in that code or codes for which a given value of \(\tau\) will minimize \(\psi(t,\omega)\) for \(\omega\neq 0\). The problem of finding such a code is closely related to the coding problem in information theory and many of the methods employed there are applicable. For example, for \(\tau\) = \(2^{-1}\) the best codes yet found (and there is reason to believe no better codes exist) are those called by various authors \(\text{AL}, \text{DP}, \text{DM}, \text{and SH})\) shift-register or null sequence codes of maximum length. An example of such a code for \(n = 5\), \(\text{AL} = 31\) is the following:

\[
01101100001110111011100000111
\]

which is obtained by starting off with the code of length 5, 01101 (any other starting code except 00000 will yield equivalent results), and setting the next element equal to the sum modulo 2 of the first, second, third, and fifth digits preceding. This process is repeated for each successive element. It will be found that after 31 elements have been written down, the sequence will repeat, and indeed, the fact that the code does not repeat prior to the \(2^n\)-1 element is a sufficient test that a proper parity check rule has been employed. For any value of \(n\) there are a number of such codes—all of which, so far has been determined, are equally satisfactory for our present purposes. It should be recalled that we are interested, loosely, in minimizing the maximum value of \(\psi(t,\omega)\) for all values of \(t\) and \(\omega\) excluding the origin. In particular it would not be sufficient alone to minimize \(\psi(t,\omega)\) along the \(\tau\) axis, i.e., for \(\omega = 0\). Codes can be found which have better performance along the \(\tau\) axis than the maximum length null sequences, but such codes inevitably seem to have large spikes, i.e., potential ambiguities, or the \(\omega\) axis. Experience would seem to suggest that maximum-length null sequence codes yield a maximum value of \(\psi(t,\omega)\) (excluding the origin) the order of \(\text{AL}\).

We are justified in concluding that—from
standpoint of accuracy and ambiguity—coded-pulse or other noise-like waveforms achieve essentially the ultimate possible performance. Why then have such waveforms not had a wider application in radar? To be sure, an appreciation of the value of such waveforms is relatively recent and there is a feeling, which we do not completely share, that the equipment to generate and process such waveforms is impractically complicated. But the most important reason is that noise waveforms have in many practical target situations serious disadvantages from the standpoint of resolution. This is a problem which we now wish to investigate in general.

5.0 Resolution

Thus far we have considered only cases in which it was known a priori that at most one target echo was present at any one time. Such a fortuitous situation almost never occurs in practice. At the very least we have to discriminate against our own transmitted signal which represents a huge signal at zero range and velocity. Moreover, in many cases ground clutter, chaff, the ionosphere, meteor trails, etc., return echoes which not only contain little useful information but which, if they have a large amplitude, may obscure the echoes from desired targets. Indeed, it is probably only a slight exaggeration to claim that in many cases the problem of resolving desired echoes from undesired echoes and from one another is so important as to be the principal requirement on the radar design. Freedom from ambiguities, for example, might be a nice thing to have, but not if it can be obtained only with a reduction in resolution performance.

Despite the importance of this problem, there is remarkably little we can say about it of any general validity of utility. There are, of course, several obvious platitudes which despite their triviality help to formulate the problem.

1. If the interfering signal is known completely and exactly (i.e., if \( k_1 = k_2 \) and the carrier phase angle are known precisely) then there is really no resolution problem since the obvious and theoretically correct procedure is to subtract a replica of the interfering signal from \( \mathbf{r}(t) \) prior to processing.

2. If the class of undesired signals is identical with the class of desired signals then there is obviously no possibility for resolution. Thus the resolution problem is theoretically and practically an interesting one only if the exact characteristics of the interfering signals are unknown in one or more respects and different in one or more respects from the desired signals. There are, of course, many kinds of situations, but a large fraction of these are of little interest. For example, two signals, which are identical except for amplitude obviously can be resolved with high reliability only if the most probable desired signal energy is much greater than the most probable undesired signal energy. The most interesting cases are those in which the signals to be distinguished differ in time delay and/or frequency shift. Here the appropriate measure of the difference between signals is

\( \psi_r(t, \nu) \) and there are two cases of interest:

1. Each desired signal is orthogonal to the entire class of undesired signals.
2. Each desired signal is at most weakly correlated with some of the members of the class of undesired signals.

Clearly, resolution is almost impossible if the undesired signals are large and strongly correlated with the desired signals.

In the first case resolution is trivially simple to obtain. It can easily be shown that the a posteriori probability that any particular desired signal is present is entirely independent of the presence or size of the undesired signals. Hence all of the analysis in preceding sections with respect to reliability of detection, accuracy, and ambiguity is immediately applicable. Resolution in this case is obtained automatically.

On the other hand, in the second case the situation is not nearly so clear. We might, of course, just pretend that the signals to be separated are orthogonal and thus build our decision circuits as previously. The resolution performance under these conditions will be quite good, i.e., \( P \) and \( P_f \) for the desired signals will be essentially unaltered by the presence of the undesired signals, provided that the ratio of the energy of the undesired signal to the energy of the desired signal remains somewhat less than \( (\frac{1}{\sqrt{2}})^2 \). Thus a degree of relative resolution can be obtained. However, it should be possible to achieve somewhat better relative resolution by altering the form of the detector, e.g., intentional using an appropriately mis-matched filter. Of course, the reliability of detection for desired signals will then be less, but in the presence of correlated undesired signals such a loss in reliability of detection is inevitable. In certain hypothetical cases the best possible form for the mis-matched filter can be worked out, for example, if the class of undesired signals consists of a finite set of signals at known discrete values of \( \tau \) and \( \nu \) but with unknown amplitude it is possible to design a detector providing essentially infinite resolution at only a (usually) small price in detectability. But for the case of continuous parameter distributions no really satisfactory procedures are known. However it seems unlikely that any truly significant improvements in performance can be achieved by mismatching the filter. Empirical methods for designing clutter rejection filters, for example, are probably as good as any.

The conclusion is inescapable that if resolution is important the radar waveform must be so chosen as to make the signals to be resolved as nearly orthogonal as possible. From this point of view the periodic pulse waveform has the existing advantages constituting as good a reason as any for its overwhelming popularity. By permitting ambiguities, the periodic pulse waveform manages to cram nearly all the volumes required under the
... surface into tall slender spikes, leaving most of the plane absolutely empty. No other waveform is quite so well suited for those applications (e.g., GCI radars) in which resolution capability is the pre-eminent design specification. However, when slight compromises can be tolerated in resolution performance, important advantages in other respects can be achieved by several variations in the coherent periodic pulse waveform, e.g., non-coherent phase from pulse-to-pulse, a periodically-repeated phase-or-frequency modulated pulse, or a time-duplexed radar employing two repetition rates, each for half the echo duration. Other schemes, e.g., staggered repetition rate, or changing frequency from pulse-to-pulse, are less useful since they have a serious effect on resolution performance.

We do not intend to create the impression that any periodic pulse radar is an adequate solution to the usual, let alone the extreme, resolution situation; it is not. There are many radar systems in particular which fail to achieve the desired performance, if for no other reason, because either the resolution performance of the periodic pulse radar is inadequate or because the ambiguities associated with this waveform are intolerable. We feel (and we believe to be) the pre-eminent design specification. However, when slight compromises can be tolerated in resolution performance, important advantages in other respects can be achieved by several variations in the coherent periodic pulse waveform, e.g., non-coherent phase from pulse-to-pulse, a periodically-repeated phase-or-frequency modulated pulse, or a time-duplexed radar employing two repetition rates, each for half the echo duration. Other schemes, e.g., staggered repetition rate, or changing frequency from pulse-to-pulse, are less useful since they have a serious effect on resolution performance.

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